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Maximizing total tardiness on a single machine in O(n^2) time via a reduction to half-product minimization

Sergey Kovalev [a]

a INSEEC Business School, 25 rue de l'Université 69007 Lyon, France

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Maximizing total tardiness on a single machine in $O(n^2)$ time via a reduction to half-product minimization

Sergey Kovalev

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Abstract Gafarov et al. [6] have recently presented an $O(n^2)$ time dynamic programming algorithm for a single machine scheduling problem to maximize the total job tardiness. We reduce this problem in $O(n \log n)$ time to a problem of unconstrained minimization of a function of 0-1 variables, called half-product, for which a simple $O(n^2)$ time dynamic programming algorithm is known in the literature.

Keywords Scheduling · Single machine · Total tardiness · Maximization · Dynamic programming

1 Introduction

Usually, cost minimization is considered as an optimization criterion in scheduling problems, with a rare exception of revenue maximization, see, for example, recent publications of Gawiejnowicz and Kononov [8], Hashemian et al. [9], and Keshavarz et al. [14] for cost minimization and Can and Ulusoy [4] and St. John and Tóth [19] for revenue maximization. However, in certain situations it makes sense to change the direction of optimization for cost objectives. Aloulou et al. [1,2] argue that solutions to the cost maximization problems provide an information about how poorly schedules can perform. This information can be used to predict consequences of uncertainty in scheduling, which can be due to an unexpected request during schedule implementation, for example, a request of completing some jobs earlier than the others.

The problem in this paper can be formulated as follows. Jobs of a set $N = \{1, \dots, n\}$ have to be processed without preemption on a single machine from time zero with no idle time between them. Job j is associated with a processing time p_j and a due date d_j, $j = 1, \dots, n$. A schedule is completely characterized by the job sequence. Let C_j denote the completion time of job j in a given schedule. *Tardiness* of job j is defined as $T_j = \max\{0, C_j - d_j\}$. The problem is to find a schedule that maximizes the total tardiness $\sum T_j$. Here and below we assume that all summations are taken over all jobs if it is not stated otherwise. This problem is denoted as $1(no - idle)|| \max \sum T_j$.

Gafarov et al. [6] give another motivation for this problem, in which the due date d_j represents the beginning of a low cost processing interval for job j. Jobs have to be processed in the interval $[0, \sum p_j]$ and the total time of their processing in the respective low cost intervals have to be maximized. Extensions of the problem $1(no - idle)|| \max \sum T_j$ to address job weights, release times and deadlines are studied by Gafarov et al. [7].

It was known since 1969 that the problem $1(no-idle)|| \max \sum w_j T_j$ with positive integer job weights w_j is pseudo-polynomially solvable by the algorithm of Lawler and Moore [18]. In [18], the no-idle time assumption is mentioned in the first paragraph of Section 4 and the maximization of the total weighted tardiness in the last paragraph of Section 8, where the fact that a the total weighted earliness of a sequence calculated with respect to due dates d_j is equal to the total

Sergey Kovalev
INSEEC Business School ECE Lyon, 19 place Tolozan, 69001 Lyon, France
Tel.: +33 (0) 4 37 92 92 44
E-mail: skovalev@inseec.com

weighted tardiness of the reverse sequence with respect to the due dates $d'_j = d_j - p_j$, $j = 1, \dots, n$, is implicitly employed.

In 2012 Gafarov et al. [6] made a breakthrough by presenting an $O(n^2)$ algorithm for the problem $1(no-idle)||\max\sum T_j$. In the next section, we reduce this problem in $O(n\log n)$ time to a problem of unconstrained minimization of a function of 0-1 variables, called half-product in the literature, for which a simple $O(n^2)$ time dynamic programming algorithm exists.

2 Reduction

Given a schedule, job j is called *on-time* if $C_j \le d_j$, and it is called *tardy* if $C_j > d_j$. Assume that the jobs are numbered in the *Longest Processing Time (LPT)* order such that $p_1 \ge \dots \ge p_n$.

We employ the following property.

Lemma 1 *There exists an optimal schedule for the problem $1(no-idle)||\max\sum T_j$ in which on-time jobs are sequenced arbitrarily and precede tardy jobs that are sequenced in the LPT order.*

Proof. The statements that tardy jobs follow on-time jobs and that they are sequenced in the LPT order in an optimal schedule are proved by the job interchange argument, see Lawler and Moore [18] and Gafarov et al. [6]. Further, consider such an optimal schedule. Let E denote the set of on-time jobs in this schedule. Observe that jobs of the set E must complete by the earliest due date in this set because otherwise job with the earliest due date in the set E can be placed last among them, its due date will be missed, and the total tardiness will increase. On the other hand, if jobs of the set E complete by the earliest due date in this set, then their order does not affect the total tardiness. $\square$

Introduce 0-1 variables x_j interpreting them so that jobs with $x_j = 0$ are on-time and jobs with $x_j = 1$ are tardy, $j = 1, \dots, n$. Denote $x = (x_1, \dots, x_n)$ and $P = \sum p_j$. Our reduction is based on the following lemma.

Lemma 2 *Let x^* be a maximizer of the function*

$$F(x) = \sum_{i=1}^{n}\Big(P - d_i - \sum_{j=i+1}^{n} p_j x_j\Big)x_i = \sum_{i=1}^{n}(P-d_i)x_i - \sum_{1\le i<j\le n} p_j x_i x_j$$

on the set of 0-1 vectors $x = (x_1, \dots, x_n)$, where $\sum_{j=n+1}^{n} p_j x_j := 0$. Schedule that satisfies Lemma 1 and in which the set of tardy jobs is defined as

$$\{i \in N \mid x_i^* = 1, P - \sum_{j=i+1}^{n} p_j x_j^* - d_i > 0\}$$

is optimal for the problem $1(no-idle)||\max\sum T_j$.

Proof. We call value $\Big(P - d_i - \sum_{j=i+1}^{n} p_j x_j\Big)x_i$ *contribution* of variable x_i to the value $F(x)$. Thus, $F(x)$ is equal to the total contribution of all variables $x_1, \dots, x_n$. In Fig. 1 these contributions are illustrated for vector x corresponding to a schedule satisfying Lemma 1, in which E and T are the sets of on-time and tardy jobs, respectively. There, δ_i is the contribution of variable $x_i = 1$ and $d_{j^0} = \min\{d_j \mid x_j = 0\}$.

Consider an optimal schedule S^0 that satisfies Lemma 1, in which T^0 is the set of tardy jobs. Let A^0 be its total tardiness. Define vector $x^0 = (x_1^0, \dots, x_n^0)$ such that $x_i^0 = 1$ if $i \in T^0$, and $x_i^0 = 0$ if $i \in N \backslash T^0$. We have $F(x^0) = A^0$. Now, consider an arbitrary maximizer x^* of the function $F(x)$. We have $F(x^*) \ge A^0$. Define job set $T^* := \{i \in N \mid x_i^* = 1, P - \sum_{j=i+1}^{n} p_j x_j^* - d_i > 0\}$. We have

$$F(x^*) \le \sum_{i\in T^*}(P - \sum_{j=i+1}^{n} p_j x_j^* - d_i). \tag{1}$$

Consider schedule S^* in which jobs of the set $N\backslash T^*$ are scheduled first in the LPT order and jobs of the set T^* are scheduled last in the LPT order. Let A^* be the total tardiness of this schedule. For $i \in T^*$, we have $\sum_{j=i+1}^{n} p_j x_j^* \ge \sum_{i+1\le j\le n, j\in T^*} p_j$. Therefore, for $i \in T^*$,

$$P - \sum_{i+1\le j\le n, j\in T^*} p_j - d_i \ge P - \sum_{j=i+1}^{n} p_j x_j^* - d_i > 0$$

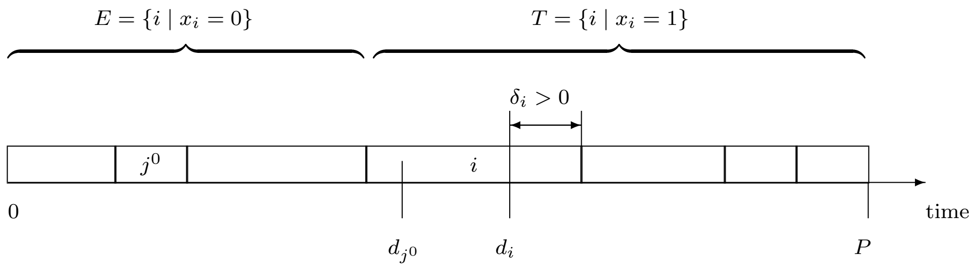

Fig. 1 Variables and their contributions to $F(x)$ for a schedule satisfying Lemma 1

and

$$A^* \geq \sum_{i \in T^*} (P - \sum_{i+1 \leq j \leq n, j \in T^*} p_j - d_i) \geq \sum_{i \in T^*} (P - \sum_{j=i+1}^{n} p_j x_j^* - d_i). \tag{2}$$

From $F(x^*) \geq A^0$, (1) and (2), we deduce $A^* \geq F(x^*) \geq A^0$. Since A^0 is the optimal value of the maximum total tardiness, we further deduce that the schedule S^* is optimal for the problem $1(no - idle)|| \max \sum T_j$.

Note that the non-strict inequalities in (2) must be satisfied as equalities because otherwise they will lead to $A^* > A^0$ which is a contradiction. Thus,

$$A^* = \sum_{i \in T^*} (P - \sum_{i+1 \leq j \leq n, j \in T^*} p_j - d_i) = \sum_{i \in T^*} (P - \sum_{j=i+1}^{n} p_j x_j^* - d_i).$$

These equalities, the definition of the set T^* and the definition of the schedule S^* imply that in S^* there is no on-time job between the tardy jobs which are scheduled after the last on-time job in the LPT order. Thus, the schedule S^* satisfies Lemma 1. □

Observe that, in order to construct the function $F(x)$, we only need the LPT numbering for the values p_j and d_j, which can be done in $O(n \log n)$ time.

Badics and Boros [3] have introduced the function

$$H(x) = \sum_{1 \leq i < j \leq n} a_i b_j x_i x_j - \sum_{i=1}^{n} c_i x_i$$

of 0-1 variables x_i, $i = 1, \ldots, n$, which they have called *half-product*. There, $(a_1, \ldots, a_{n-1})$ and $(b_2, \ldots, b_n)$ are vectors of non-negative integers and $(c_1, \ldots, c_n)$ is an arbitrary integer vector. They proved that minimizing the half-product on the set of 0-1 vectors is NP-hard and gave a fully polynomial time approximation scheme (FPTAS), which delivers a solution with a given relative error ε, $0 < \varepsilon \leq 1$, in $O(n^2 \log \sum a_i / \varepsilon)$ time. Jurisch et al. [11] presented a simple optimization dynamic programming algorithm with the run time $O(n \sum a_i)$ to minimize the half-product $H(x)$.

We note that function $-F(x)$ is an example of the half-product, with $c_i = d_i - P$, $i = 1, \ldots, n$, $a_i = 1$, $i = 1, \ldots, n-1$, and $b_j = p_j$, $j = 2, \ldots, n$. Hence, it can be minimized in $O(n^2)$ time. Since minimizing $-F(x)$ is equivalent to maximizing $F(x)$, we deduce that the problem $1(no - idle)|| \max \sum T_j$ can be solved in $O(n^2)$ time by the algorithm of Jurisch et al. [11].

There exists a number of results on the half-product minimization problem, its special cases and its generalizations, see, for example, Kubiak [16,17], Jurisch et al. [11], Janiak et al. [10], Erel and Ghosh [5], Xu [20], Kellerer and Strusevich [12,13] and Kovalyov and Kubiak [15]. Several scheduling problems reduce to the half-product minimization. Problem $1(no - idle)|| \max \sum T_j$ extends this list.

3 Conclusions and suggestions for future research

We reduced the problem $1(no{-}idle)|| \max \sum T_j$ to a problem of unconstrained minimization of the function $-F(x)$ of 0-1 variables, which is a special case of the half-product minimization problem and admits a simple $O(n^2)$ dynamic programming algorithm. This approach is an alternative to the algorithm of Gafarov et al. [6] with the same theoretical run time estimation.

It is interesting to know if the problem $1(no-idle)||\max\sum T_j$ can be solved in $O(n\log n)$ time. Our intuition is that it is possible. While at present we do not know how to do it, an idea is to use the fact that, in an optimal solution, contributions of variables $x_i = 1$ are positive, which is equivalent to $\sum_{j=1}^{i-1} x_j \le A_i$ if $x_i = 1$, where $A_i = (P-d_i)/p_i - 1$ if $(P-d_i)/p_i$ is integer and $A_i = \left\lfloor (P-d_i)/p_i \right\rfloor$ if $(P-d_i)/p_i$ is not integer.

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