Working Paper
n°: 2014-53-01

Portfolio insurance: Gap risk under conditional multiples

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An ulterior version of this article appeared in *European Journal of Operational Research*, Volume 236, Issue 1, July 2014, Pages 238–253.
It can be purchased at: doi:10.1016/j.ejor.2013.11.027

Portfolio insurance: Gap risk under conditional multiples

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This version: February 2012

The research on financial portfolio optimization has been originally developed by Markowitz (1952). It has been further extended in many directions, among them the portfolio insurance theory introduced by Leland and Rubinstein (1976) for the "Option Based Portfolio Insurance" (OBPI) and Perold (1986) for the "Constant Proportion Portfolio Insurance" method (CPPI). The recent financial crisis has dramatically emphasized the interest of such portfolio strategies. This paper examines the CPPI method when the multiple is allowed to vary over time. To control the risk of such portfolio management, a quantile approach is introduced together with expected shortfall criteria. In this framework, we provide explicit upper bounds on the multiple as function of past asset returns and volatilities. These values can be statistically estimated from financial data, using for example ARCH type models. We show how the multiple can be chosen in order to satisfy the guarantee condition, at a given level of probability and for various financial market conditions.

Keywords: Finance; Risk management; Portfolio insurance; CPPI; Conditional multiple.

Introduction

Portfolio selection theory has been originally introduced by Markowitz (1952) and further developped in many directions (see e.g. Merton (1971, 1990), for the continuous time setting). Various decision criteria such as the expected utility theory can be considered to determine the optimal portfolios (see e.g. Campbell and Viceira, 2002; Prigent, 2007; Yu *et al.* 2009). The impact of portfolio rebalancing methods has been also investigated (see e.g. Detemple et al., 2003; Yu and Lee, 2011; Wang and Forsyth, 2011). However, investors oftens searchs for additional guarantees (portfolio insurance), in particular when financial markets drop. The financial management industry extensively uses portfolio insurance methods for various financial instruments: equities, bonds, structured credit products, hedge funds...

Portfolio insurance has two main objectives: first, to allow investors to recover at maturity at least a given percentage of his inital investment (usually 100%); second, to benefit from potential financial market rises. Thus, it allows investors to limit downside risk in bearish financial market, second to benefit from bullish markets.Such portfolio

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management is dramatically relevant during financial crisis. The two main portfolio insurance methods are: the Option Based Portfolio Insurance (OBPI), introduced by Leland and Rubinstein (1976); the Constant Proportion Portfolio Insurance (CPPI) considered by Perold (1986) for fixed-income instruments and Black and Jones (1987) for equity instruments (see also Perold and Sharpe, 1988). As regards OBPI method, the portfolio is invested in a risky benchmark asset covered by a put option written on it. The strike of the option is equal to a fixed proportion of the initially invested amount (which corresponds to the capital insured at maturity).

The CPPI method is based on a dynamic portfolio allocation along the whole management period. The investor determines a floor which is defined as the lowest acceptable portfolio value. Then, he invests a amount ("the exposure") in the risky asset, which corresponds to a given proportion (called "the multiple") of the excess of the portfolio value over the floor (this difference is usually called "the cushion"). The remaining funds are usually invested in cash (treasury-bills for example). Both the management parameters (floor and multiple) depend on the investor's risk tolerance. The CPPI strategy implies that the exposure is about zero if the cushion value is near zero. In continuous-time, this property prevents portfolio value from falling below the floor, except if a very sharp drop in the market occurs before the investor can modify his portfolio allocation. Some of the properties of portfolio insurance have been previously studied by Black and Rouhani (1989) and Black and Perold (1992) when the risky asset follows a geometric Brownian motion (GBM) and by Bertrand and Prigent (2003) when the volatility is stochastic. Comparisons between standard portfolio insurance methods are illustrated by Bookstaber and Langsam (2000), Cesari and Cremonini (2003) and Bertrand and Prigent (2005, 2011). The main conclusion is that it is not so easy to rank these strategies, except by their sensitivity Vega to the volatility of the risky asset. However, the CPPI method is the best strategy when the market drops or increases by a significant amount.[3]

The main issue of the CPPI strategy is to choose the crucial parameter which determines the portfolio risk exposure, known as the multiple. Note that financial institutions directly bear the risk of the insured portfolios they sell: at maturity, if the portfolio value is smaller than the guaranteed floor ("the gap risk"), they must compensate the corresponding loss with their own capital. Thus, one crucial question for the financial institution that promotes such funds is: what exposure to the risky asset or, equivalently, what level of the multiple to accept? On one hand, since the portfolio expected return is increasing with respect to the multiple, customers want the multiple as high as possible. On the other hand, due to market imperfections[4], portfolio managers

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[3]Note also that, using various stochastic dominance (SD) criteria up to third order and assuming that the risky underlying asset follows a GBM, Zagst and Kraus (2011) provide very specific parameter conditions implying the second- and third-order SD of the CPPI strategy.

[4]For example, portfolio managers cannot actually rebalance portfolios in continuous

must control the gap risk by setting the multiple smaller than an upper bound. If a maximal daily historical drop (e.g. -20%) is anticipated during the time period, the portfolio manager chooses $m \leq 5$. It implies low expected portfolio returns. If the risky asset drop is assumed to be less significant (e.g. -10%), the upper bound can be chosen higher (e.g. $m \leq 10$). If the portfolio manager wants to use higher multiple values, he can based the strategy on quantile hedging. In that case, the multiple can be chosen as high as possible but so that the portfolio value will always be above the floor at a given probability level (typically 99%).

The answer to this latter issue has important practical implications. However, the usual assumption in the literature is that the multiple is constant through time. It implies that: it has to be determined initially; it is unconditional, meaning that, whatever the market and portfolio value fluctuations, the risk exposure is always equal to the same proportion of the cushion. Thus, such portfolio strategy does not take sufficiently account of the risk that the risky asset drops, which challenges the concept of dynamic portfolio insurance. In a discrete-time setting, the extreme value approach has been applied to the standard CPPI method with constant multiple by Bertrand and Prigent (2002) to control the gap risk. Balder *et al.* (2009) have analyzed CPPI effectiveness using quantile conditions when the risky asset follows a geometric Brownian motion (GBM) that is discretized at deterministic times. Cont and Tankov (2009) have examined CPPI strategies for exponential Lévy processes. But all these unconditional methods reduce the risk exposure to a constant risky asset exposure, which cannot dynamically adjusted.

In this paper, we introduce another CPPI method, directly linked to a risk management approach and based on a conditional multiple. In this setting, we determine upper bounds on the conditional time-varying multiple, using both quantile ("Value-at-Risk") and expected shortfall criteria to control gap risks. Such downside risk measures have been introduced in the late ninety and further analyzed (see e.g. Pedersen and Satchell, 1998; Artzner *et al.*, 1999; Szegö, 2002). They are related to economical capital allocation as recommanded by Basel II for banking laws and regulations (see Goovaerts *et al.*, 2002). They have been also intensively used in portfolio management (see Rockafellar and Uryasev, 2002). Unlike Hamidi *et al.* (2009a, b) who use conditional autogressive Value-at-Risk to estimate gap risks, we consider a quite general parametric model based on ARCH type return modelling. In this framework, we succeed in identifying exactly the various upper bounds. Our results prove that a conditional multiple can be determined as functions of state variables. These latter ones are usually the past stock logretruns and volatilities.

The remainder of the paper is organized as follows. Section 2 presents the basic properties of the CPPI model. Section 3 introduces the modified CPPI method with a

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time.

conditional multiple, based on various quantile and expected shortfall conditions. In particular, upper bounds on the multiple are provided and analyzed. Section 4 illustrates numerically the previous theoretical results and provides a detailed description of the portfolio return distributions. Some of the proofs are gathered in Appendix.

The standard CPPI model

The financial market

Basically, two assets are involved: the riskless asset, B, with a constant interest rate r (usually Treasury bills or other liquid money market instruments) and the risky one, S (usually a market index, equity index for instance). Changes in asset prices are supposed to occur at discrete times $(t_k)_{1\leq k\leq n}$ along a whole management period $[0,T]$.

The riskless asset evolves according to deterministic rates denoted by r_{t_k}. The variations of the stock price S between two times t_{k-1} and t_k are defined by:

$$\Delta S_{t_k} = S_{t_k} - S_{t_{k-1}}.$$

Since we search an upper bound on the multiple m, we have to focus on the left hand side of the probability distribution of $\frac{\Delta S_{t_k}}{S_{t_{k-1}}}$. Thus, we introduce the notation :

$$X_{t_k} = -\frac{\Delta S_{t_k}}{S_{t_{k-1}}} = \frac{S_{t_{k-1}} - S_{t_k}}{S_{t_{k-1}}},$$

where X_{t_k} denotes the opposite of the relative jump of the risky asset at time t_k. In fact, when we want to determine an upper bound on the multiple m, we have only to consider positive values of X.

Denote by $M_T,$ the maximum of the finite sequence $(X_{t_k})_{1\leq k\leq n}$. We have:

$$M_T = Max(X_{t_1},\ldots,X_{t_n}).$$

The standard CPPI portfolio

Usually, the investor, having initially invested an amount V_0, wants to recover a fixed percentage p of V_0 at a given maturity T. To provide a terminal portfolio value V_T higher than the insured amount pV_0, the portfolio manager keeps the portfolio value V_{t_k} above the floor $F_{t_k} = p.V_0.e^{-r(T-t_k)}$ at any time t_k during the management period $[0,T]$. For this purpose, he determines:

- The floor F_{t_k} which must be lower than the portfolio value V_{t_k} at any time t_k.
- The amount e_{t_k} (also called *the exposure)* invested in the risky asset, which is a fixed proportion m of *the cushion* C_{t_k} that is the excess of the portfolio value

over the floor.

- The constant m, usually called *the multiple,* which is is a nonnegative constant.

The remaining fund $(V_{t_k} - e_{t_k})$ is invested in the riskless asset with a deterministic rate r_{t_k} for the period $[t_{k-1}, t_k]$. Since $C_t = V_t - F_t$, this insurance method consists in keeping C_t positive at any time t in the period.
Both the floor and the multiple are functions of the investor's risk tolerance. The higher the multiple, the more the investor will benefit from increases in stock prices. Nevertheless, the higher the multiple, the faster the portfolio will approach the floor when there is a sustained decrease in stock prices. As the cushion approaches zero, exposure approaches zero, too. In continuous-time, when asset dynamics have is no jump, this keeps portfolio value from falling below the floor. Nevertheless, during financial crises a very sharp drop in the market may occur before the manager has a chance to trade. This implies that m must not be too high (for example, if a fall of 10% occurs, m must not be greater than 10 in order to keep the cushion positive).

Denote by V_{t_k} the value of the portfolio at time t_k. As explained in the introduction, the CPPI method is based on the following portfolio insurance condition:

- There exists a deterministic floor F_{t_k} such that at any time t_k, the value V_{t_k} must be above the floor.
- The total amount (also called the exposition) e_{t_k} invested on the underlying asset S is equal to mC_{t_k} where the cushion C_{t_k} represents the difference $V_{t_k} - F_{t_k}$ between the portfolio value and the floor.
- The multiple m is a nonnegative constant.
- The remaining amount is invested on the riskless asset with a deterministic rate r_{t_k} for the period $[t_{k-1}, t_k]$.

The higher the multiple m, the higher the amount invested on the risky asset. Therefore, a speculative investor would choose high values for m. Nevertheless, in that case, his portfolio is riskier and, as shown in what follows, his guarantee may no longer hold. Indeed, we easily deduce that the portfolio value is solution of

$$V_{t_k} = V_{t_{k-1}} - e_{t_{k-1}} X_{t_k} + (V_{t_{k-1}} - e_{t_{k-1}}) r_{t_k},$$

from which, we deduce the cushion dynamics:

$$C_{t_k} = C_{t_{k-1}} \left[1 - m\, X_{t_k} + (1-m) r_{t_k} \right].$$

Since, for all dates $t_k,$ the cushion must be positive, we get finally the condition: for all $k \leq n$,

$$-mX_{t_k} + (1-m) r_{t_k} \geq -1.$$

In fact, since r_{t_k} is relatively small, the previous inequality yields to the following relation that gives an upper bound on the multiple:

Proposition *The guarantee is satisfied at any time of the management period with a probability equal to* 1 *if and only if*

$$\forall k \leq n, X_{t_k} \leq \frac{1}{m} \text{ or equivalently } M_T = Max(X_{t_k})_{k \leq n} \leq \frac{1}{m}.$$

Since the right end point d *of the common distribution* F *of the variables* X_{t_k} *is positive, we deduce that the insurance is perfect along any period* $[0,T]$ *if and only if* m *is smaller than* $\frac{1}{d}$ *.* For example, if the maximal drop is equal to 20% , then $d = 0,2$. Thus m must be less than 5 .

Quantile conditions

The previous condition, which is rather strong, can be modified if a quantile hedging approach is adopted, like for the Value-at-Risk (see Föllmer and Leukert (1999) for application of this notion in financial modelling). For example, the new condition is to guarantee that the portfolio value will be always above the floor at a given probability level $1 - \epsilon.$ This gives the following relation for a period $[0,T]$:

$$\mathsf{P}[C_{t_k} \geq 0, \forall t_k \in [0,T]] \geq 1 - \epsilon,$$

or, equivalently, if M_T denotes the maximum of the X_{t_k} for times t_k in $[0,T]$:

$$\mathsf{P}[\forall t_k \in [0,T], X_{t_k} \leq \frac{1}{m}] = \mathsf{P}[M_T \leq \frac{1}{m}] \geq 1 - \epsilon.$$

Note that, since m is nonnegative, the condition $X_{t_k} \leq \frac{1}{m}$ is equivalent to $\left[X_{t_k} \mathrm{I}_{\{0 \leq X_{t_k}\}}\right] \leq \frac{1}{m}$. It means that this condition is stringent only when the risky return drops.

Then we can detail the quantile hedging condition. For this, introduce the function $F_{M_T}^{-1}$ defined as the inverse of the distribution function F_{M_T} which is assumed to be strictly increasing. Then, we get (see Prigent, 2001):

Proposition *For small* $r_k,$ *we have (approximately) the following condition:*

$$m \leq \frac{1}{F_{M_T}^{-1}((1-\epsilon))}.$$

Additionally, if the sequence $(X_k)_k$ *is i.i.d. with common cdf* F *, then we have:*

$$m \leq \frac{1}{F^{-1}\left((1-\epsilon)^{\frac{1}{T}}\right)}.$$

This condition gives an upper limit on the multiple m which is obviously higher than the standard limit $\frac{1}{d}$.

CPPI with a conditional multiple

Conditional multiple

The simplicity and flexibility of the CPPI method allows the introduction of several extensions:

- First, we can introduce a stochastic floor, in particular to keep past profits from rises in the stock market. For example, Estep and Kritzman (1988) have introduced the Time Invariant Portfolio Protection (TIPP). This method is based on the following guarantee condition:
- $$V_{t_k} \geq g \times \max(F_{t_k}, \sup_{l \leq k} V_{t_l}),$$
- where g is an exogenous parameter which lies between 0 and 1. In this case, the investor does not want to lose more than a given percentage of the maximum of his past portfolio values. This strategy is of CPPI type but with a stochastic floor, given by $g \times \max(F_{t_k}, \sup_{l \leq k} V_{t_l})$.
- Second, the multiple can be no longer fixed but determined from market and portfolio dynamics:
  - For instance, Prigent (2001) introduces a more general exposure function $e(t, C)$ defined on $[0, T] \times \mathbb{R}^+,$ positive and continuous.
  - This new exposure e has the following form:
  - $$e_{t_k} = e(t_k, C_{t_k}).$$
  - Then, conditions on $e(t_k, C_{t_k})$ must be imposed such that the cushion always is positive.
  - 1) If the cushion is nil, then the exposure must be equal to 0 :
  - $$e(t_k, 0) = 0.$$
  - 2) If the relative sizes of jumps $\frac{\Delta S}{S}$ have a lower bound d (negative), then, for any $(t_k, C_{t_k}),$ we must have:
  - $$e(t_k, C_{t_k}) \leq -\frac{1}{d} C_{t_k}.$$
  -
- In what follows, we examine the problem of the determination of a conditional multiple, using a quantile condition. To illustrate our approach, we assume that the risky asset logreturn follows an Arch type model.

Quantile hedging and conditional multiple

We allow now the CPPI multiple to vary at times t_k . Therefore, we consider a sequence of multiple values $(m_{t_k})_k$. The multiple is assumed to be constant (equal to $m_{t_{k-1}}$) on the time period $[t_{k-1}, t_k[$. Our purpose is to provide explicit conditions on the sequence of multiples $(m_{t_k})_k$ to control the gap risk. This later one can induce failure at the level of CPPI strategy: indeed, after a sharp drop in the risky asset price S , the portfolio value can break through the floor so that the cushion becomes negative. This risk can be controlled by adjusting the multiple value by means of a quantile criterion corresponding to a Value-at-Risk approach. Most of previous studies introduce a standard geometric Brownian motion to model the risky aset dynamics. Although this process is widely used in financial theory, it does not take account of potential jumps nor any temporal dependence of financial returns. Portfolio rebalancing is also asssumed to be in continous-time, which is not exactly the usual practice. To overcome these drawbacks, it is possible to use discrete-time processes such as the huge familly of ARCH models that take particularly the dependence between returns into account.

General quantile conditions

To control the gap risk of the CPPI portfolio for any time period $[t_{k-1}, t_k[$, we introduce a sequence of positive thresholds $(L_{t_{k-1}})_k$ for potential downside cushion values. The idea is to control various probabilities that the cushions become smaller than the thresholds. Since the CPPI strategy implies that the cushion must be nonnegative at any time t_{k-1} , then the thresholds $L_{t_{k-1}}$ must also be positive. In any case, condition $C_{t_k} > L_{t_{k-1}}$ implies the usual guarantee constraint $C_{t_k} > 0$.

Remark *(Choice of the thresholds) The value of the threshold* $L_{t_{k-1}}$ *at time* t_{k-1} *can depend on the risk tolerance of the investor or be imposed by specific guarantee constraints. For example, we can set :*

1) $L_{t_{k-1}} = 0$ *. It corresponds to the standard insurance constraint:* $C_{t_k} > 0$ *, which means that the portfolio value must be above the floor.*

2) $L_{t_{k-1}} = L$ *constant. We can choose for example a constant value for* L *as a given proportion of the initial cushion* $L = qC_{t_0}$ *.*[5]

3) $L_{t_{k-1}} = qC_{t_{k-1}},$ *where* q *is a fixed proportion of the cushion values. In that case, we search to get* $C_{t_k} > qC_{t_{k-1}}$ *. In this case the threshold is adjusted to the value of the cushion at any time* t_{k-1} *. Setting* $q > 1$ *or* $q < 1$ *depends of risk tolerance.*

A first global quantile condition is defined by:

$$\mathrm{P}[\forall t_k, C_{t_k} > L_{t_{k-1}}] \geq 1 - \epsilon.$$

This condition can be deduced from other mild assumptions:

- Assumption ($A1$):

$$\forall k, \mathrm{P}[C_{t_k} > L_{t_{k-1}} | C_{t_1} > L_{t_0}, \ldots, C_{t_{k-1}} > L_{t_{k-2}}] \geq (1-\epsilon)^{1/T}.$$

[5]In practice, CPPI strategies are defined for an average period of 5 year. Then, if we choose this threshold L at the beginning of the period, it may be too restrictive or obsolete thereafter.

- Assumption ($A2$):

  $\forall k, \mathsf{P}^{\mathsf{F}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] \geq (1-\epsilon)^{1/T}.$

- Assumption ($A3$):

  $\forall k, \mathsf{P}^{\mathsf{G}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] \geq (1-\epsilon)^{1/T},$

- where $\mathsf{F}_{t_{k-1}}$ denotes the $\sigma-$ algebra generated by the random variables X_{t_1} ,..., X_{t_k} and $\mathsf{G}_{t_{k-1}}$ is the $\sigma-$ algebra generated by $\mathsf{F}_{t_{k-1}}$ and the random event $\{C_{t_{k-1}} > 0\}$. Recall that $\mathsf{G}_{t_{k-1}}$ is defined as the smallest $\sigma-$ algebra that contains all intersections of any subset of $\mathsf{F}_{t_{k-1}}$ with the event $\{C_{t_{k-1}} > 0\}$.

Proposition *Each of these three assumptions implies Property (GuaranteeQuantile).*
Proof - Suppose ($A1$) is satisfied. Then, using the equality:

$$\mathsf{P}[C_{t_1} > L_{t_0}, \ldots, C_T > L_{t_{n-1}}] =$$

$$\mathsf{P}[C_{t_1} > L_{t_0}] \times \mathsf{P}[C_{t_2} > L_{t_1} | C_{t_1} > L_{t_0}] \times \ldots \times \mathsf{P}[C_T > L_{t_{n-1}} | C_{t_1} > L_{t_0}, \ldots . C_{t_{n-1}} > L_{t_{n-2}}],$$

from which, we deduce the result.
- Suppose now that ($A2$) is satisfied. For any event A , denote by $\mathsf{P}[A|Y]$ the probability that A occurs knowing Y ($\mathsf{P}[A|Y]$ is defined as the conditional expectation: $\mathsf{E}[\mathsf{I}_A|Y]$).
We use the following lemma: for any random variables X and Y,

$$\mathsf{P}[X < L|Y] \geq a \Rightarrow \mathsf{P}[X < L|Y \in B] \geq a, \forall B \subset \mathbb{R}^{k-1}.$$

Let:

$$X = C_{t_k} \text{ and } Y = (C_{t_1}, \ldots, C_{t_{k-1}}).$$

Introduce the set B defined by:

$$B = \left\{(c_{t_1}, \ldots, c_{t_{k-1}}) \in \mathsf{R}^{k-1} \text{ with } c_{t_1} > L_{t_0}, \ldots, c_{t_{k-1}} > L_{t_{k-2}}\right\}.$$

Then, we have:

$$\forall k, \mathsf{P}[C_{t_k} > L_{t_{k-1}} | C_{t_1} > L_{t_0}, \ldots, C_{t_{k-1}} > L_{t_{k-2}}] \geq (1-\epsilon)^{1/T},$$

from which, we deduce that assumption (A_1) is satisfied. This implies Property (GuaranteeQuantile).
- Condition ($A3$) implies condition ($A1$) since $\mathsf{G}_{t_{k-1}}$ contains the event $C_{t_1} > L_{t_0} \cap \ldots \cap C_{t_{k-1}} > L_{t_{k-2}}$. Thus, it also implies Property (GuaranteeQuantile).

In what follows, we focus on Condition (A_3): $\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] \geq (1-\epsilon)^{1/T}$. Since we consider small time periods (for example, daily rebalancing), we can assume to simplify that $r_{t_k} = 0$. Recall that the cushion value is determined from the following relation:

$$C_{t_k} = C_{t_{k-1}} \times (1 - m_{t_{k-1}} \times X_{t_k}).$$

The guarantee condition at any time t_k is: $C_{t_k} > L_{t_{k-1}}$.

Case 1: ($C_{t_{k-1}} > 0$)

$$C_{t_k} > L_{t_{k-1}} \Leftrightarrow 1 - m_{t_{k-1}} \times X_{t_k} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}.$$

Case 2: ($C_{t_{k-1}} < 0$)

$$C_{t_k} > L_{t_{k-1}} \Leftrightarrow 1 - m_{t_{k-1}} \times X_{t_k} < \frac{L_{t_{k-1}}}{C_{t_{k-1}}}.$$

Note that, in this latter case, we must stop usually the investment on the risky asset. Then, for all the remaining time periods, the whole portfolio value is invested on the riskless asset.

Therefore, at any time t_{k-1}, the value of the multiple $m_{t_{k-1}}$ must be chosen such that, at time t_k, the cushion value satisfies: $C_{t_k} > L_{t_{k-1}}$. Consequently, $m_{t_{k-1}}$ can be searched as the following function:

$$m_{t_{k-1}} = g_{t_{k-1}} \times \mathrm{I}_{C_{t_{k-1}}>0} + h_{t_{k-1}} \times \mathrm{I}_{C_{t_{k-1}}<0},$$

where both $g_{t_{k-1}}$ and $h_{t_{k-1}}$ are random variables which are $\mathcal{F}_{t_{k-1}}-$ measurable. Since we must stop the investment on the risky asset as soon as the cushion is nul, we must set $h_{t_{k-1}} = 0$.

Determination of the multiple for the quantile condition

In what follows, we search for explicit forms of the random variables $g_{t_{k-1}}$. We assume that the risky asset logreturn Y follows a Garch(p,q) model. As it is well-known (see e.g. Gourieroux, 1997), this kind of dynamics is quite suitable to describe asset fluctuations in a discrete-time setting. The ARCH (*Autoregressive Conditionally Heteroscedastic*) models, introduced by Engle (1982), are specific non-linear time series models. They can describe quite exhaustive set of the underlying dynamics. They have been largely applied on macroeconomics and statistical theory.

The GARCH model is defined as follows. The logreturn Y is defined by:

$$Y_{t_k} = \ln\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right) \Leftrightarrow \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} = \exp(Y_{t_k}) - 1.$$

Consider the system of auto regressive equations:

$$\begin{cases} Y_{t_k} = \alpha_0 + \sum_{i=1}^{p} \alpha_i \times Y_{t_{k-i}} + \sigma_{t_k} \times \varepsilon_{t_k}, \\ \Lambda(\sigma_{t_k}) = c_0 + \sum_{j=1}^{q} c_j(\varepsilon_{t_{k-j}}) \times d_1 \Lambda(\sigma_{t_{k-1}}), \end{cases}$$

where σ_{t_k} denotes the volatility, the sequence $(\varepsilon_{t_k})_k$ is i.i.d with common pdf $f_\varepsilon > 0$ and Λ , c_0 and d_1 are constant, and $c_j(.)$ are deterministic functions. The function $\Lambda : \mathbb{R}^+ \to \mathbb{R}$ is assumed to be strictly increasing. Note that the volatility σ_{t_k} is known at time t_{k-1} . It means that the sequence $(\sigma_{t_k})_k$ is predictable.

The information delivered by the observation of risk asset returns until time t_{k-1} is generated by the random vector $(\varepsilon_{t_1}, \ldots, \varepsilon_{t_{k-1}})_k$.

We have:

$$\mathsf{F}_{t_{k-1}} = \sigma - \text{algebra } (\varepsilon_{t_1}, \ldots, \varepsilon_{t_{k-1}}).$$

Thus, the random variables $g_{t_{k-1}}$ are deterministic functions of $(\varepsilon_{t_1}, \ldots, \varepsilon_{t_{k-1}})$.

Therefore, we have to search for multiples $m_{t_{k-1}}$ which have the following form:

$$m_{t_{k-1}} = g(t_{k-1}, Y_{t_1}, \ldots, Y_{t_{k-1}}, \sigma_{t_1}, \ldots, \sigma_{t_k}) \times \mathrm{I}_{C_{t_{k-1}} > 0}.$$

Remark *Note also that the random variables $Y_{t_{k-1}}$ are deterministic functions of random vectors $(\varepsilon_{t_1}, \ldots, \varepsilon_{t_{k-1}})$. But, for a better financial interpretation of the multiple, we explicitly introduce functions of vectors $(Y_{t_1}, \ldots, Y_{t_{k-1}})$ themselves. Indeed, we examine how the conditional multiple depends on both the volatility levels and on the past logreturns according to given criteria based on specific risk measures. These two kinds of variables are here the state variables.*

In order to determine the conditional multiple, three cases have to be distinguished:

1) The cushion at time t_{k-1} satisfies: $C_{t_{k-1}} > L_{t_{k-1}} (> 0)$.
2) The cushion at time t_{k-1} satisfies: $0 < C_{t_{k-1}} < L_{t_{k-1}}$.
3) The cushion at time t_{k-1} is non positive: $C_{t_{k-1}} < 0$ (in that case, we do no longer invest on the risky asset until the maturity).

Denote:

$$Z_{t_{k-1}} = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_{t_k} \times F^{-1}(1-(1-\epsilon)^{1/T}),$$

where F denotes the common cdf of ε_{t_k} .[6] Note that $F^{-1}(1-(1-\epsilon)^{1/T})$ is a particular value of the random variable ε_k , since it is its quantile at the level $(1-(1-\epsilon)^{1/T})$). Therefore, for small values of ϵ , $F^{-1}(1-(1-\epsilon)^{1/T})$ is negative. It implies that the logreturn value $Z_{t_{k-1}}$ is a decreasing function with respect to the volatility σ_{t_k} .

We get the following result (see proof in Appendix).

Proposition *The quantile condition ($A3$) at time t_{k-1} can be satisfied according to the following conditions:*

- Case 1: $C_{t_{k-1}} > L_{t_{k-1}}$:

- (1-1) If $\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 < 0,$ *then* $\mathrm{P}^{\mathcal{G}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] = 1$ *. It means that if the cushion at time* t_{k-1} *is relatively high, and if the investor chooses a multiple that is not too high, then his portfolio is always guaranteed against market fluctuations.*

- (1-2) If $\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 > 0,$ *then:*

- (1-2-i) If $Z_{t_{k-1}} < 0$ *, the conditional multiple must satisfy:*

$$m_{t_{k-1}} \leq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp[Z_{t_{k-1}}] - 1}.$$

- (1-2-ii) if $Z_{t_{k-1}} > 0,$ *there is no constraint for* $m_{t_{k-1}}$ *(recall that we take* $m_{t_{k-1}} > 0$ *). If the investor wants to use a higher multiple then this one must be smaller than a given upper bound if logreturn value* $Z_{t_{k-1}}$ *is negative (condition 1-2-i: "bear market"). If the*

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[6]If we assume as usual that the pdf of the random variables ε is strictly positive, then the cdf F is invertible. Otherwise, we can consider the generalized inverse for monotone functions.

marker is rather bullish (condition 1-2-ii), then the multiple is not bounded.

- Case 2: $(0 <)C_{t_{k-1}} < L_{t_{k-1}}$: *Then*

- (2-i) If $Z_{t_{k-1}} < 0$ *, there is no solution for* $m_{t_{k-1}}$. *.*

- (2-ii) if $Z_{t_{k-1}} > 0$ *, the conditional multiple must satisfy:*

$$m_{t_{k-1}} > \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp[Z_{t_{k-1}}] - 1}.$$

Here, since the cuhion value at time t_{k-1} *is relatively small, the multiple must be sufficiently high in order to get a cushion higher than the threshold at time* t_k *.*

Corollary *For* $L_{t_{k-1}} = 0$ *, the quantile condition (* $A3$ *) at time* t_{k-1} *is characterized by:*

- If condition (1-1) $\frac{-1}{m_{t_{k-1}}} + 1 < 0$ *is met, then* $\mathrm{P}^{G_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] = 1$ *. But this condition means that the multiple is smaller than* $1,$ *which is not appropriate if the investor wants to make benefit from market rises.*

i) If condition (1-2) $\frac{-1}{m_{t_{k-1}}} + 1 > 0$ *is met, then :*

If $Z_{t_{k-1}} < 0$ *(condition 1-2-i), the multiple must be upper bounded:*

$$m_{t_{k-1}} \leq \frac{1}{1 - \exp[Z_{t_{k-1}}]}.$$

ii) If $Z_{t_{k-1}} > 0$ *(condition 1-2-ii), then there is no constraint on* $m_{t_{k-1}}$.

Remark *The previous result shows that, if at time* t_{k-1} *, the auto regressive terms* $\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}$ *and the conditional volatility* σ_{t_k} *are sufficiently high, then the logreturn* Y_{t_k} *has a sufficiently high probability to be positive and thus, there is no constraint on the multiple at time* t_{k-1} [7] *.*

Remark *When the cushion is positive at time* $t_{k-1},$ *the choice of the multiple is very flexible. Thus, within the quantile condition at time* $t_{k-1},$ *we can add some other conditions on the multiple to better benefit from market conditions. When the cushion is negative (which happens with a small probability), the quantile condition generally cannot be satisfied, except for small values of* $(1-\epsilon).$ *But, in this case, this is not a true insurance condition. Therefore, a possible strategy is to adopt the previous condition when the cushion is positive and to invest the whole portfolio value on the riskless asset, as soon as the cushion is negative.*

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[7] Note that $F^{-1}(1-(1-\epsilon)^{1/T})$ is a particular value of the random variable ϵ_k (since it is the quantile at the level $(1-(1-\epsilon)^{1/T})$).

Expected shortfall conditions

In what follows, we consider several expected shortfalls (ES) criteria under the dependence logreturn condition. Note that Balder *et al.* (2009) examine expected shortfall criterion but they only consider the case with i.i.d Gaussian logreturns and fixed multiple m (see also Cont and Tankov, for i.i.d Lévy logreturns and also fixed multiple m). The quantile measure (VaR criterion) is the maximum loss not exceeded with a given probability defined at the confidence level α . A major drawback of the quantile approach is that it does not provide any information about the severity of losses exceeding the quantile value. Therefore, the expected shortfall (also called "Conditional Value-at-Risk") can be an alternative to better take account of the average loss beyond the quantile value.

The three criteria to choose the multiple variable, using ES approach:

Three possible criteria based on ES condition can be considered depending on the information level (recall that $\mathcal{G}_{t_{k-1}}$ is the $\sigma-$ algebra generated by $\mathcal{F}_{t_{k-1}}$ and the random event $\{C_{t_{k-1}} > 0\}$.):

- Criterion B1: $\min_{m_{t_{k-1}}} \mathbb{E}[L_{t_{k-1}} - C_{t_k} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}]$;
- Criterion B2: $\min_{m_{t_{k-1}}} \mathbb{E}[L_{t_{k-1}} - C_{t_k} | \mathcal{F}_{t_{k-1}} \cap C_{t_k} < L_{t_{k-1}}]$;
- Criterion B3: $\min_{m_{t_{k-1}}} \mathbb{E}[L_{t_{k-1}} - C_{t_k} | \mathcal{G}_{t_{k-1}} \cap C_{t_k} < L_{t_{k-1}}]$.

However, the first two criteria are not exactly appropriated:

- Criterion B1 leads to solve the following minimization problem:

$$\min_{m_{t_{k-1}}} \mathbb{E}[L_{t_{k-1}} - C_{t_k} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}].$$

We have:

$$\mathbb{E}[L_{t_{k-1}} - C_{t_k} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}]$$

$$= \mathbb{E}[L_{t_{k-1}} - C_{t_k}(1 + m_{t_{k-1}}[e^{Y_{t_k}} - 1]) | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}],$$
$$= L_{t_{k-1}} - C_{t_{k-1}}(1 + m_{t_{k-1}} \mathbb{E}[e^{Y_{t_k}} - 1 | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}])$$

Then:

- If $\mathbb{E}[e^{Y_{t_k}} - 1 | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}] > 0,$ then there is no solution (this is an ill-posed problem since its solution corresponds to - ∞). It corresponds to the search for the maximal conditional expectation of the cushion.

- If $\mathsf{E}[e^{Y_{t_k}} - 1 | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}}] < 0,$ then $m_{t_{k-1}} = 0$, which is a degenerate solution.

- Criterion B2 does not take account of the constraint on the portfolio strategy if the cushion becomes negative (i.e. $m_{t_{k-1}} = 0$ if $C_{t_{k-1}} \leq 0$).

Therefore, we use only the criterion B3 that we detail and analyze in what follows.

Study of the criterion B3

Recall that the risk control of a financial position X by means of expected shortfall criterion is based on two steps:

-First, we have to determine the Value-at-Risk $VaR_X(\alpha)$ at a given probability level $\alpha.$ We have:

$$\mathsf{P}[X \leq -VaR_X(\alpha)] = \alpha,\ \alpha\epsilon]0,1[.$$

Note that $VaR_X(\alpha)$ is equal to the opposite of the quantile $q_X(\alpha)$ of X at the level α .

- Second, we calculate the expected shortfall defined by:

$$ES_X(\alpha) = \mathsf{E}[q_X(\alpha) - X | X \leq q_X(\alpha)] = \alpha,\ \alpha\epsilon]0,1[.$$

We apply a similar approach for the study of criterion B3 in a dynamic framework:

- First, we determine the values of $m_{t_{k-1}}$ which correspond to the quantile condition as defined in Assumption A3. For the given probability level α ($= 1 - (1-\epsilon)^{\frac{1}{T}}$ in assumption A3), we consider the set $M_{t_{k-1}}(\alpha, L_{t_{k-1}})$ of $m_{t_{k-1}}$ values such that:

$$\mathsf{P}[C_{t_k} \leq L_{t_{k-1}} | \mathsf{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0] \leq \alpha,\ \alpha\epsilon]0,1[.$$

Note that the set $M_{t_{k-1}}(\alpha, L_{t_{k-1}})$ is determined according to Proposition (PropositionQuantileMultiple).

- Then, the control of the expected shortfall is defined on the previous set from criterion B3:

$$\min_{m_{t_{k-1}} \in M_{t_{k-1}}(\alpha, L_{t_{k-1}})} L_{t_{k-1}} - \mathsf{E}[C_{t_k} | \mathsf{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} \leq L_{t_{k-1}}].$$

Examine now the term $L_{t_{k-1}} - \mathsf{E}[C_{t_k} | \mathsf{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} \leq L_{t_{k-1}}]$. Recall that we consider $L_{t_{k-1}} \geq 0$. Since we have:

$$C_{t_k} = C_{t_{k-1}}[1 + m_{t_{k-1}}(\exp(Y_{t_k}) - 1)].$$

Then we deduce:

$$\mathsf{E}[L_{t_{k-1}} - C_{t_k} | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}} \cap C_{t_k} < L_{t_{k-1}}] =$$

$$\frac{\mathsf{E}[(L_{t_{k-1}} - C_{t_k}) \mathsf{I}_{C_{t_k} < L_{t_{k-1}}} | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}}]}{\mathsf{P}[C_{t_k} < L_{t_{k-1}} | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}}]},$$

where I_A , is the Heaviside function associated to the event A defined here by " C_{t_k} is smaller than $L_{t_{k-1}}$."

Denote $\Phi_L(m_{t_{k-1}})$ the function given by:

$$\Phi_L(m_{t_i}) = \frac{\mathsf{E}[(L_{t_{k-1}} - C_{t_k}) \mathsf{I}_{C_{t_k} < L_{t_{k-1}}} | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}}]}{\mathsf{P}[C_{t_k} < L_{t_{k-1}} | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}}]}.$$

<u>Determination of $\Phi_L(m_{t_{k-1}})$:</u>

Since $C_{t_k} = C_{t_{k-1}}[1 + m_{t_{k-1}}(\exp(Y_{t_k}) - 1)]$ and $C_{t_{k-1}} > 0$, then we have the equivalence:

$$C_{t_k} < L_{t_{k-1}} \Leftrightarrow [1 + m_{t_{k-1}}(\exp(Y_{t_k}) - 1)] < \frac{L_{t_{k-1}}}{C_{t_{k-1}}}.$$

Recall that:

$$Y_{t_k} = a_{t_{k-1}} + b_{t_{k-1}} \times \varepsilon_{t_k},$$

where $a_{t_{k-1}}$ and $b_{t_{k-1}}$ are defined by

$$\begin{cases} a_{t_{k-1}} = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}, \\ b_{t_{k-1}} = \sigma_{t_k} > 0. \end{cases}$$

Therefore, the condition $[1 + m_{t_{k-1}}(\exp(Y_{t_k}) - 1)] < L_{t_{k-1}}$ is equivalent to:

$$\varepsilon_{t_k} < \frac{\ln\left[\left(\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1\right)/m_{t_{k-1}} + 1\right] - a_{t_{k-1}}}{b_{t_{k-1}}}.$$

Denote $l_L(m_{t_{k-1}}) = \frac{\ln\left[\left(\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1\right)/m_{t_{k-1}} + 1\right] - a_{t_{k-1}}}{b_{t_{k-1}}}$. Then, we deduce:

Proposition *The expected shortfall condition corresponds to the conditional expectation*

$$\mathrm{E}[L_{t_{k-1}} - C_{t_k} | \mathrm{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} < L_{t_{k-1}}]$$

which is equal to

$$L_{t_{k-1}} - \frac{C_{t_{k-1}} \int_{-\infty}^{l_L(m_{t_{k-1}})} [1 + m_{t_{k-1}}(\exp(a_{t_{k-1}} + b_{t_{k-1}} x) - 1)] \times f_\varepsilon(x) dx}{\int_{-\infty}^{l_L(m_{t_{k-1}})} f_\varepsilon(x) dx}.$$

Suppose for example that $\varepsilon_{t_{i+1}}$ *is Gaussian.*

Corollary *The expected shortfall condition for the Gaussian case corresponds to the conditional expectation: (See Appendix 2)*

$$\Phi_L(m_{t_{k-1}}) = L_{t_{k-1}} - C_{t_{k-1}}[1 - m_{t_{k-1}}(1 - \exp(\frac{b_{t_{k-1}}^2 + 2a_{t_{k-1}}}{2})\frac{N(l_L(m_{t_{k-1}}) - b_{t_{k-1}})}{N(l_L(m_{t_{k-1}}))})],$$

where

$$\begin{cases} l_L(m_{t_{k-1}}) = \frac{\ln\left[\left(\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1\right)/m_{t_{k-1}} + 1\right] - a_{t_{k-1}}}{b_{t_{k-1}}}, \\ l_L(m_{t_{k-1}}) - b_{t_{k-1}} = \frac{\ln\left[\left(\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1\right)/m_{t_{k-1}} + 1\right] - a_{t_{k-1}} - b_{t_{k-1}}^2}{b_{t_{k-1}}}. \end{cases}$$

<u>Value of $\Phi_L(m_{t_{k-1}})$ for $L_{t_{k-1}} = 0$:</u>

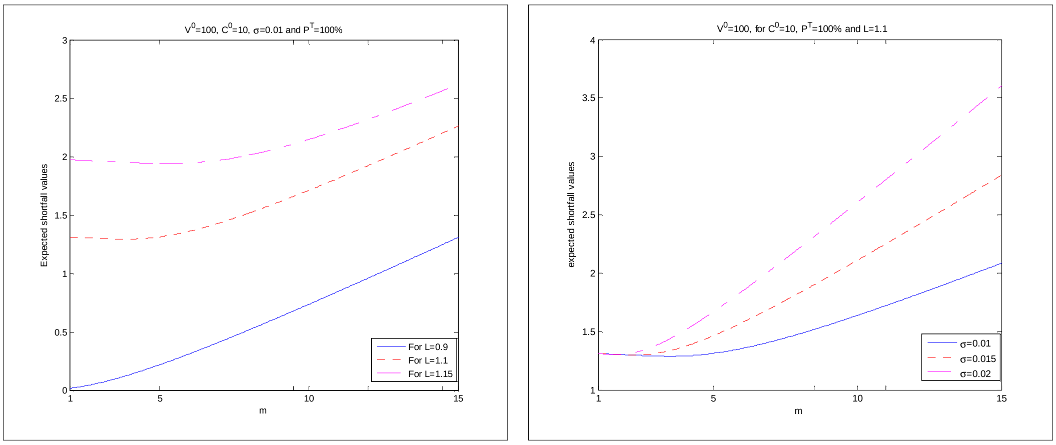
Note that, for the standard case $L_{t_{k-1}} = 0$, we have:

$$\begin{cases} l_0(m_{t_{k-1}}) = \dfrac{\ln(1-\frac{1}{m_{t_{k-1}}})-a_{t_{k-1}}}{b_{t_{k-1}}}, \\ l_0(m_{t_{k-1}}) - b_{t_{k-1}} = \dfrac{\ln(1-\frac{1}{m_{t_{k-1}}})-a_{t_{k-1}}-b^2_{t_{k-1}}}{b_{t_{k-1}}}, \end{cases}$$

where $l_0(m_{t_{k-1}})$ does not depend on the cushion value $C_{t_{k-1}}$ at time t_{k-1}. Therefore, we get:

$$\Phi_0(m_{t_{k-1}}) = -C_{t_{k-1}}\left[1 - m_{t_{k-1}}\left(1 - \exp(\frac{b^2_{t_{k-1}} + 2a_{t_{k-1}}}{2})\frac{N(l_0(m_{t_{k-1}}) - b_{t_{k-1}})}{N(l_0(m_{t_{k-1}}))}\right)\right].$$

This function can be illustrated as follows.



In Figure esf simul fonction de m, we simulate the value of the expected shortfall function Φ_0 for several thresholds L and volatilities σ , for $V_0 = 100$ and $C_0 = 10$. The expected shortfall value can reach a (interior) minimum between the lower and upper bounds. The multiple value at which the minimum is reached decreases when the volatility increases. It means that the fund manager must reduce the risk exposure when the usual volatility risk rises. On the contrary, the multiple value increases when the level floor increases. An interpration of this feature is that, if the protection due to the floor is more efficient, then we can expose more the remainding fund invested on the risky asset. If we choose $L_{t_{k-1}} < C_{t_{k-1}}$ ($L = 0.9$ in Figure esf simul fonction de m,), the minimization of the expected shortfall $\Phi_0(m_{t_{k-1}})$ gives always a value of the multiple equal to the lower bound. In some sense, for $L_{t_{k-1}} > C_{t_{k-1}}$, the fund manager must increase the risk exposure in order to expand the probability that $L_{t_{k-1}} < C_{t_k}$.

In all cases, the conditional volatility and the floor level play a crucial role to determine the conditional multiple that minimizes the expected shortfall function at each time during the managment period. Note that this function provides an explicit relation between the risk aversion of the investor when he chooses the floor level, the estimation of financial market parameters through the conditional volatility and finally the management parameter, namely the conditional multiple $m_{t_{k-1}}$.

Minimization according to criterion B3

To summarize, for a fixed probability level α, we have to solve successively:

First step: determination of the solutions corresponding to the quantile condition (VaR of the cushion)

$$\mathsf{P}[C_{t_k} \leq L_{t_{k-1}} | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}}] \leq \alpha.$$

with

$$\begin{cases} C_{t_k} = C_{t_{k-1}}[1 + m_{t_{k-1}}(\exp(Y_{t_k}) - 1)], \\ Y_{t_k} = a_{t_{k-1}} + b_{t_{k-1}} \times \varepsilon_k. \end{cases}$$

From Subsection (Subsection Quantile), we have:

Consider a given level $L_{t_{k-1}}$. The condition $C_k \leq L_{t_{k-1}}$ is equivalent to:

$$\varepsilon_{t_k} \leq \underbrace{\frac{\ln[\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1] - a_{t_{k-1}}}{b_{t_{k-1}}}}_{q_{\alpha(\varepsilon)}}.$$

Then, the condition

$$\mathsf{P}[C_{t_k} \leq L_{t_{k-1}} | C_{t_{k-1}} > 0 \cap \mathsf{F}_{t_{k-1}}] \leq \alpha,$$

determines the multiple variable $m_{t_{k-1}}$ for each level of α and fixed $L_{t_{k-1}}$ (See section 2, criterion A3):

- If $C_{t_{k-1}} > L_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 < 0$:

$$m_{t_{k-1}} \leq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1}.$$

- If $C_{t_{k-1}} > L_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 > 0$ (i.e. $Z_{t_{k-1}} > 0$), no constraint on $m_{t_{k-1}}$.
- If $C_{t_{k-1}} < L_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 < 0$ (i.e. $Z_{t_{k-1}} < 0$), no solution.

When $L_{t_{k-1}} = 0$, we get:

$$m_{t_{k-1}} \leq \frac{1}{1 - \exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)})}.$$

- If ($0 <$) $C_{t_{k-1}} < L_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 > 0$:

$$m_{t_{k-1}} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1}.$$

Denote $\widetilde{m}_{t_{k-1}}(\alpha, L_{t_{k-1}}) = \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1}$ and $M_{t_{k-1}}(\alpha, L_{t_{k-1}})$ the set of conditional multiples $m_{t_{k-1}}$ satisfying the four previous conditions.

Second Step: minimization of the expected shortfall

$$\widetilde{\mathrm{ES}}_{\alpha}[C_{t_k}] =$$

$$Min_{\{m_{t_{k-1}} \in M_{t_{k-1}}(\alpha, L_{t_{k-1}})\}} (L_{t_{k-1}} - \mathrm{E}[C_{t_k} | \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} \leq L_{t_{k-1}}]).$$

The expected shortfall is minimized for values of the conditional multiple $m_{t_{k-1}}$ which lie between the two bounds of the multiple variable $m_{t_{k-1}}$ and satisfy the VaR condition.

Simulations

We illustrate numerically the values of the CPPI portfolios for both constant and conditional multiples in two main cases: the risky logreturns of S are independent; the risky logreturns of S are dependent according to an EGARCH model.

First case: i.i.d. returns and fixed multiple

For the i.i.d. case, we can determine the value of the multiple according to independence assumption on logreturn process.

For example, suppose that S_t follows a geometric Brownian motion, which is the most commonly used financial model. We have:

$$S_t = S_0 \times \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma \times \sqrt{t} B_t),$$

where, in particular, B_t has a standard Gaussian distribution $N(0,1)$. Denote by N its cdf.

In that case, the random $Z_{t_{k-1}}$ defined in Relation (Definition Z) is constant:

$$Z_{t_{k-1}} = (\mu - \frac{1}{2} \times \sigma^2)\Delta t + \sigma\sqrt{\Delta t} \times N^{-1}(1 - (1 - \epsilon)^{1/T}).$$

Proposition *Since the fixed multiple* m *is positive, then the probability that there exists a date* t_k *such that the cushion* C_{t_k} *is negative is given by:*

$$1 - \mathsf{P}[\forall t_k, C_{t_k} > 0] = 1 - \left(1 - F\left(\ln\left[1 - \frac{1}{m}\right]\right)\right)^T,$$

with

$$F(x) = N\left[\frac{x - (\mu - \frac{1}{2} \times \sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right].$$

For the simulation ,we consider a standard CPPI strategy based on various maket timing rebalancing. The time horizon is set to 5 years. We illustrate in particular the correspondence between the level of the probability $(1-\epsilon)$ to get the guarantee, the multiple m and the standard deviation σ . For the numerical base case, we fix the value of the instantaneous rate of return $\mu = 5\%$ and consider a volatility equal to 20% . Table (probabilty default) provides the probability that the guarantee does not hold during the management period, according to rebalancing frequency, management parameter (here, the multiple m) and financial parameters μ and σ .

$\sigma = 20\%, \quad \mu = 5\%$

	daily rebalancing	weekly rebalancing	monthly rebalancing	annual rebalancing
$m = 5$	0	0	9.5380×10^{-5}	0.5067
$m = 7$	0	3.0800×10^{-11}	0.0158	0.7109
$m = 9$	0	2.3408×10^{-7}	0.1290	0.8027

$m = 7 \ , \quad \mu = 5\%$

	daily rebalancing	weekly rebalancing	monthly rebalancing	annual rebalancing
$\sigma = 15\%$	0	0	1.0690×10^{-4}	0.5243
$\sigma = 20\%$	0	3.0800×10^{-11}	0.0158	0.7109
$\sigma = 25\%$	0	6.2147×10^{-7}	0.1603	0.8108

$m = 7 \ , \quad \sigma = 20\%$

	daily rebalancing	weekly rebalancing	monthly rebalancing	annual rebalancing

$\mu = 5\%$	0	3.0800×10^{-11}	0.0158	0.7109
$\mu = 7\%$	0	1.1546×10^{-13}	0.0042	0.6542
$\mu = 10\%$	0	0	4.4742×10^{-4}	0.5645

The rebalancing frequency plays obviously an important role to determine the probability of not having the guarantee. The higher this frequency, the smaller the probability of not having the guarantee. When the volatility or/and the constant multiple increase, the probability of not having the guarantee increases too.

Dependent returns (ARCH type models)

Many previous studies have emphasized the importance of the asymmetric model GARCH to estimate the conditional volatility. The asymmetric GARCH implies that negative shocks induce greater volatility than positive shocks. Poon and Granger (2003) compare several studies that concern the GARCH models. They conclude that in general the asymmetric volatility performs better than usual GARCH. Heynen et al. 1994) argue that exponential GARCH gives the best description of asset prices according to the Akaike information critrion. Chen and Kuan (2002) test several models to determine the conditional volatility and accept only the EGARCH for several index prices. Engel and Ng (1993) show that the model EGARCH can capture most of the asymmetry of the time series but the model presents high conditional variance. Awartani and Corradi (2005) study the daily data of the S&P 500 composite index. They prove that the asymmetric model GARCH gives the best estimation since it can captured the leverage effect. Due to the previous empirical results, we choose the EGARCH (1,1) model to illustrate the theoretical results of Section 3. We use parameter values such as in Nelson (1990) to simulate and estimate the conditional volatility. Thus, we have:

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \sigma_t \varepsilon_t,$$

where ε_t are i.i.d. variables with standard Gaussian distribution and σ_t is the conditional volatility defined by:

$$\ln(\sigma_t^2) = c_0 + g(\varepsilon_{t-1}) + d_1 \ln(\sigma_{t-1}^2),$$

with

$$g(\varepsilon_{t-1}) = \xi \times \varepsilon_{t-1} + \gamma(|\varepsilon_{t-1}| - E|\varepsilon_{t-1}|).$$

Estimation of the model parameters

We determine the parameters of the EGARCH (1,1) model from the pseudo maximum likelihood method using MATLAB. In what follows, we detail the methodology. Then, we put these parameters into the EGARCH (1,1) model to produce simulations. This method allows to get simulations very close to the true behavior of the S&P 500 historical weekly logreturns on the period 01/1970-11/2011. We have 2217 observations. The sample is split into two periods: the first one (until 11/2006) is used to estimate the

parameters; the rest of the sample is used to test the model. Next figure presents the empirical distribution function of the S&P500 weekly logreturns together with the innovations (process ε_t), the conditional standard deviation σ_t and the returns. We observe that empirical and simulated distributions are very close.
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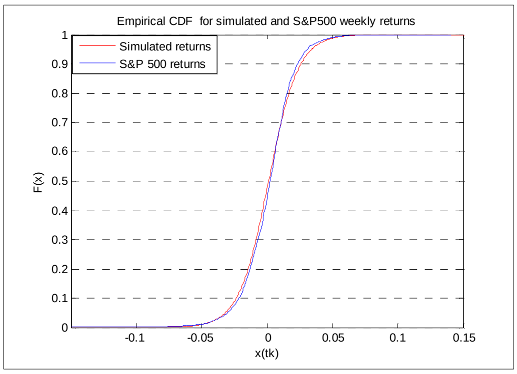


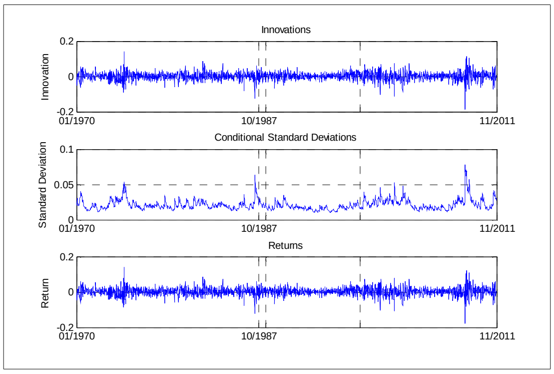


Table (descriptive statistics) presents the descriptive statistics of the S&P 500 weekly logreturns on the period 01/1970-11/2011.

Mean	maximum	minimum	std deviation	skewness	kurtosis
0.0014	0.1412	-0.1820	0.0229	-0.3184	7.6140

The S&P500 yields present an excess kurtosis compared to the Gaussian distribution $(7.6140 > 3)$. This feature illustrates the fat tails of the probability distribution that shows a significant probability of extreme returns. Note also that the skewness is negative (-0.3184) . Therefore, the S&P500 logreturn does not follow a Gaussian distribution. We observe that we have a maximum drop equal to -16% , during the financial crisis.

To determine the variable multiple, we must estimate the parameters of $Z_{t_{k-1}}$. For this purpose, we use the pseudo maximum Log-likelihood method to fit as best as possible the S&P500 data. It is shown that in this case the estimators converge (for more details on this method, see Gourieroux and Montfort,1989).This method is based on the normality assumption of conditional distribution:

$$(Y_t - \alpha_0)\backslash(\alpha_1 Y_{t-1}) \rightarrow N(0;\sigma_t).$$

The pseudo Log likelihood function with heteroscedastic errors is defined by:

$$l(\theta) = \sum_{i=1}^{n} \log f(Y_i;\theta),$$

where θ denotes the vector of parameters to be estimated.
We obtain the truncated log-likelihood function:

$$\log l(\theta) = -T\log\widehat{\sigma}_t(\theta) - \frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\frac{[Y_t - m_t(\theta)]^2}{\widehat{\sigma^2}_t(\theta)}.$$

The method of pseudo maximum likelihood estimates $\widehat{\theta}$ by finding the value of this parameters that maximize $l(\theta)$. We use an algorithms of optimization[8]
We use the EGARCH(1,1) model to estimate the conditional volatility , in the section 4-2 we have developed the criteria of choice of this model We find :

$$\log \sigma_{t_k}^2 = \underset{-0.38776}{c_0} + \underset{0.94974}{d_1} \log\sigma_{t_{k-1}}^2 + \underset{0.20676}{\gamma} [|\varepsilon_{t_{k-1}}| - \mathrm{E}\{|\varepsilon_{t_{k-1}}|\}] + \underset{-0.1214}{\xi} \varepsilon_{t_{k-1}},$$

where $\mathrm{E}\{|\varepsilon_{t_{k-1}}|\} = \sqrt{\frac{2}{\pi}}$ for the Gaussian law.
Then the conditional quantile $Z_{t_{k-1}}$ is given by:

$$Z_{t_{k-1}} = \underset{0.0013139}{\alpha_0} + \underset{-0.046059}{\alpha_1} \times Y_{t_{k-i}} + \sigma_{t_k} \times F^{-1}(1-(1-\epsilon)^{1/T}).$$

All terms are statistically significant, in particular the leverage parameter ξ which is negative. This emphasizes the importance of asymmetrical effect.
Next figure shows the paths of $Z_{t_{k-1}}$, those of he conditional multiple determined from the quantile condtion and finally those of the conditional volatility. Note that in periods of high volatility, the value of $Z_{t_{k-1}}$ is adjusted upward.

[8]We use matlab with numerical procedures to determine the parameters that minimize the function $-l(\alpha)$.

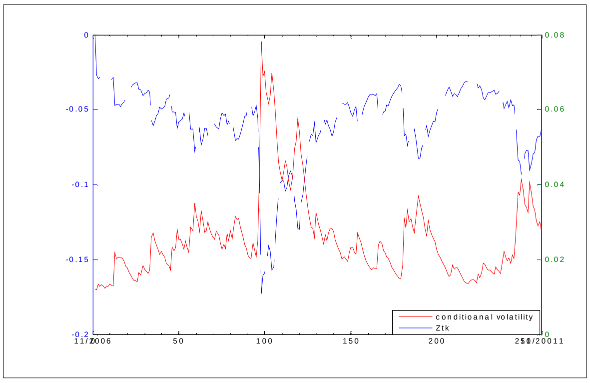


Figure A

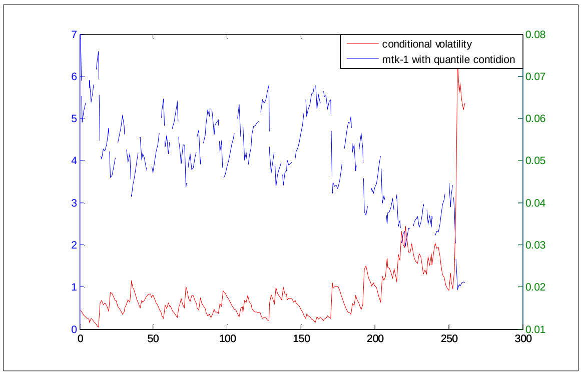


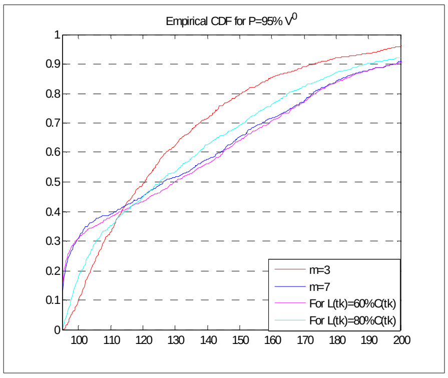
Figure B

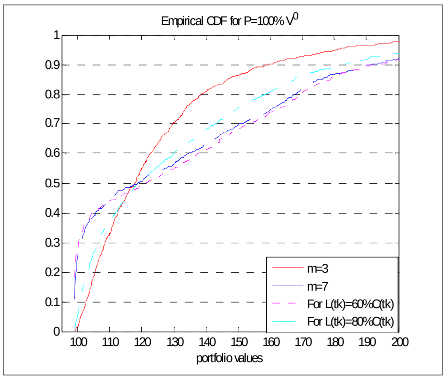
Application of the conditional multiple with quantile condition

We are interested in finding a variable multiple. As shown in Section 3, we determine the values of $\exp[Z_{t_{k-1}}]-1$ according to the EGARCH (1,1) model.
According to Proposition (PropositionQuantileMultiple) and Corollary (Corollary L=0), the variable multiple is equal to:

$$m_{t_{k-1}} = \begin{cases} \frac{1}{1-\exp[Z_{t_{k-1}}]}, \text{ for } L = 0 \\ \frac{\frac{L}{C_{t_{k-1}}}-1}{\exp[Z_{t_{k-1}}]-1}, \text{ for } L > 0 \end{cases}$$

For $L = 0$, we find that the the variable multiple values belong to the interval $[30, 60]$. Applying these values leads to too much risk exposed positions: even with small negative returns, the floor is burst almost surely (for $m = 50$, it happens as soon as the return is equal to -2%).
If we take $L > 0$ where the value of L is a given percentage of the initial cushion C_{t_0} or a proportion of the current cushion $C_{t_{k-1}}$, we obtain variable multiple values lying between 2 and 10. To illustrate this strategy, we compare its results to the standard CPPI strategy with constant multiple $m = 3$ and $m = 7$. We choose $L_{t_k} = 60\% C_{t_k}$ or $L_{t_k} = 80\% C_{t_k}$. We generate 10000 trajectories of the risky asset for a five years horizon. They correspond to the weekly logreturns from the model EGARCH (1,1). We set $V_{t_0} = 100$ at the beginning of the period and consider two guaranteed percentages $p_T = 100\% V_{t_0}$ and $p_T = 95\% V_{t_0}$ at the end of the period.
We obtain the empirical cdf of the portfolio values at maturity (five years):





We do not observe stochastic dominance of the first order between the empirical cdf for $p_T = 100$ and $p_T = 95$. The empirical cdf corresponding to the model with variable multiple and treshold $L_{t_k} = 80\% C_{t_k}$ provides a better result than the CPPI strategy with $m = 7$ and $L_{t_k} = 60\% C_{t_k}$ in the interval $[100, 120]$ of values of the simulated S&P 500. It realizes more performance than the CPPI strategy with $m = 3$ in the interval $[120, 200]$.

We remain under a parametric model, we determine a Variable Multiple according to another criterion the expected shortfall criterion, the information available at the moment t_{k-1} generated by the market volatility and the latest value of the cushion $C_{t_{k-1}}$. In the next section we show how we apply the model of the variable multiple from the new criterion.

Application of the conditional multiple with expected shortfall condition

For this criterion B3 we try to determine a multiple variable that we call $m^*_{t_{k-1}}$ we operate in two step.

The first step of this criterion, we determine a values of the multiple variable according to $C_{t_{k-1}}$ and $\exp(a + b \times q_{\alpha(\epsilon)}) - 1$ whether they are positive or negative to determine the multiple variable $m_{t_{k-1}}$ we focus for level of confidence $\alpha = 2.5\%$ and fixed $L = 10\%\, C_0$ [9]:

[9]See section 2, criterion A3.

-If $C_{t_{k-1}} > L_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 < 0$:

$$m_{t_{k-1}} \leq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1}.$$

- If $C_{t_{k-1}} > L_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 > 0$ (i.e. $Z_{t_{k-1}} > 0$), no constraint on $m_{t_{k-1}}$.
- If $C_{t_{k-1}} < L_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 < 0$ (i.e. $Z_{t_{k-1}} < 0$), no solution.

When $L_{t_{k-1}} = 0$, we get:

$$m_{t_{k-1}} \leq \frac{1}{1 - \exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)})}.$$

- If ($0 <$) $C_{t_{k-1}}$ and $\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1 > 0$:

$$m_{t_{k-1}} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp(a_{t_{k-1}} + b_{t_{k-1}} \times q_{\alpha(\varepsilon)}) - 1}.$$

for the second step we determinate a value of $m^*_{t_{k-1}}$ that minimize this expression of the expected loss

$$\widetilde{\mathrm{ES}}_\alpha[C_{t_k}] = Min_{\{m_{t_{k-1}} \in M_{t_{k-1}}(\alpha,q)\}} (L - \mathrm{E}[C_{t_k} | \mathrm{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} \leq L])$$

The interval is dynamic at every moment, the lower bound is fixed $\underline{m} = 2$ and the upper bound is the conditional value determined from the VaR condition. the threshold L_{t_k} is the same for the multiple determined from the VaR condition than from the expected shortfall condition. Then necessarily $0 \leq L \leq C_{t_{k-1}}$.

The minimization of the $\widetilde{\mathrm{ES}}_\alpha[C_{t_k}]$ always gives values of multiple equal to the lower bound.

Case 1 : $L = 0$

We have to examine the values of $\Phi_L(m_{t_{k-1}})$ as defined in Relation (PhiL). It is defined by:

$$\Phi_0(m_{t_{k-1}}) = -C_{t_{k-1}}\left[1 - m_{t_{k-1}}\left(1 - \exp(\frac{b^2+2a}{2})\frac{N(l_0(m_{t_{k-1}})-b)}{N(l_0(m_{t_{k-1}}))}\right)\right].$$

Our goal is to determine the multiple value m that minimizes this function, allowing to reduce the gap risk. The minimization of the expected shortfall criterion provides multiple values which belong to the interval $[1.5;4.5]$. These values can be applied in practice. Figure (cdf simul ESF) shows the empirical distribution functions of the payoff of the variable multiples model for $L = 0$ and for standard multiple for $m = 3$ and $m = 6$.

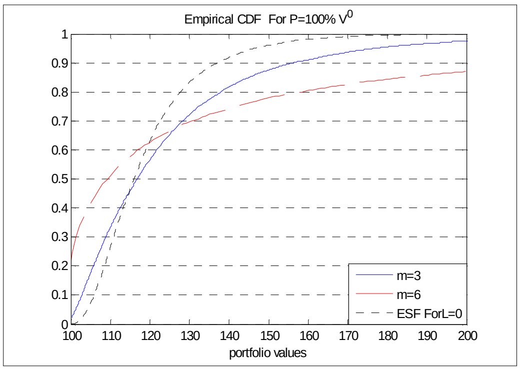


Figure A: Empirical CPPI cdf for $L = 0$.

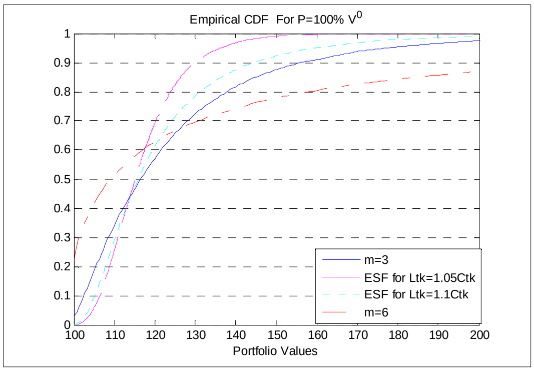


Figure B: Empirical CPPI cdf for $L > C_{t_{k-1}}$.

The fixation of a threshold $L = 0$ is too restrictive for the model under expected shortfall condition. Certainly, we can have a better guarantee than with standard mutilple for $m = 3$ and $m = 6$, but this guarantee has a high cost. It does not take full advantage of market increases.

Case 2 : $0 < L < C_{t_{k-1}}$

choosing a fixed or variable threshold L between 0 and the last cushion value allow us to determine a variable multiple that minimize the function $\Phi_L(m_{t_{k-1}})$ on a fixed interval. Numerical illustration of this model show that the minimization of the expected shortfall gives a value of multiple always equal to the lower bound of our interval.

Case 3 : $L > C_{t_{k-1}}$

This strategy is seen as a particular strategy of the CPPI method. Indeed, by fixing a thershold L_{t_k} at each moment t_k that is higher than the last cushion value $C_{t_{k-1}}$ we can then determine the value of the conditional multiple. This last one allows us to reach the particular threshold L_{t_k} while minimizing the potential loss for a level probability α.
In a previous section, we used the risk measure expected shortfall at every rebalancing moment to determine a function of potential loss that is dependent on the multiple value.
Then, this strategy with $L >$ $C_{t_{k-1}}$ can be considered as a special case of speculative strategy. The first goal of the CPPI method is to guarantee the portfolio for a fixed floor level.
To observe the values of the conditional multiple for $L >$ $C_{t_{k-1}}$, we choose the following values for $L_{t_k} = 1.05C_{t_{k-1}}$ and $L_{t_k} = 1.1C_{t_{k-1}}$ the conditional multiple m_{t_k} is the value that minimize the expected shortfall function $\Phi_L(m_{t_{k-1}})$ on a fixed interval $[1, 20]$.
This minimization allows us to have values which are applicable in practice and depend on market volatility,on the last cushion value and on the probability level that we set.
The following graph show the empirical distribution functions of the standard CPPI strategy and with conditional multiple under expected shortfall condition :
In the figurecdf simul ESF(A) we observe that if we increase the proportion of the threshold of $L_{t_k} = 1.05C_{t_{k-1}}$ at $L_{t_k} = 1.1C_{t_{k-1}}$ we increase our probabilty of capturing more market performance, we expose ourselves at the same time to more risk.

Application of the variable multiple to the S&P500:

We apply our model for the very volatile period from 11/2006 to 11/2011 we adopt a weekly rebalancing, we compare the result to the original CPPI model for $m = 3$ and $m = 6.$ During this period the market goes through several phases and particularly towards the end where we see the stock market crash related to the subprime crisis. We suppose that the log-returns of the S&P500 are dependent and follows an $EGARCH(1, 1)$ model. The selection criteria of this model are outlined in a previous section.
In the next figure, the dynamique of these portfolios for $P_T = 100\%V_0$ and for $P_T = 95\%V_0$.

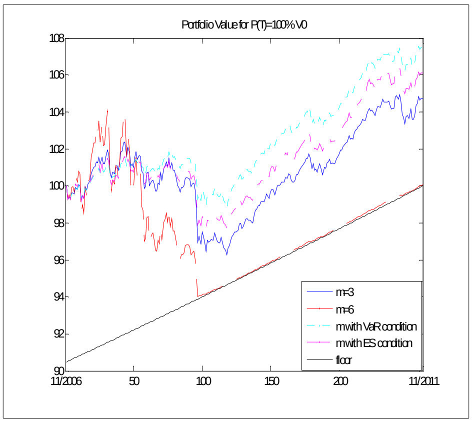


Figure A

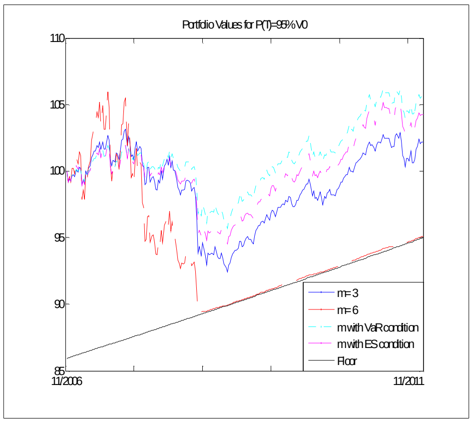


Figure B

The CPPI model with $m = 6$ gives a biggest gains than other models when the market is bullish, but with the financial crash of October 2008 the portfolio is fully monetized and falls bellow the floor at the end of the period for the two floor values $P_T = 100\% V_0$ and $P_T = 95\% V_0$, although this value of multiple is not considered a very high value. On the other hand with $m = 3,$ we were unable to capture large market performance. With the two models of the conditional multiple under VaR condition with a variable threshold $L_{t_k} = 85\% C_{t_{k-1}}$ and under the measure risk expected shortfall with $L = 1.1 C_{t_{k-1}}$ we can have higher performance than with standard multiple $m = 3.$ The muliple determined with expected shortfall condition take values in an interval $\left[1{,}5 \;\; 4{,}5\right]$, while the multiple determined with quantile condition take values in an interval $\left[1{,}5 \;\; 6\right],$ this allow to explain that portfolio CPPI with VaR condition reach a higher level than portfolio with expceted shortfall condition. Then, with the crisis subpimes and higher volatility on the market, these two models have reduced the exposure to the risky asset $S\&P500$ by decreasing the multiple values. These models remains above the floor where the other models with more exposure are dropped below.

Conclusion

As shown in this paper, it is possible to choose variable multiples for the CPPI method if quantile hedging is used and in the case of dependent logreturns. Upper bounds can be calculated for each level of probability and according to state variables. This new multiple can be determined according to the distributions of the risky asset logreturn and volatility, with the B3 criterion we want at every moment (t_{k-1}) .minimizes the loss which we can undergo,this will allow us to determine the adequate value of $(m^*_{t_{k-1}})$ and to take a risk higher. Other conditions can be imposed on this multiple, while the quantile and expected shortfall hedging constraints are satisfied. The difference with the standard multiple is significant. Other state variables can also be considered, such as exogenous macro economic factors. Finally, the impact of transaction costs can also be examined.

Appendix 1. Proof of Proposition PropositionQuantileMultiple.

In what follows, we assume that $C_{t_{k-1}} > 0$. Otherwise, as mentioned in Remark RemarqueMultipleGeneral, the total portfolio value is invested on the riskless asset.

The quantile condition ($A3$) is the following:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[1 + m_{t_{k-1}}\frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}] \geq (1-\epsilon)^{1/T},$$

which is equivalent to:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1] \geq (1-\epsilon)^{1/T}.$$

At time t_k , we have two cases:

$$\begin{cases} \exp(Y_{t_k}) < 1 : S_{t_k} < S_{t_{k-1}} \ (S \text{ decreases}), \\ \exp(Y_{t_k}) > 1 : S_{t_k} > S_{t_{k-1}} \ (S \text{ increases}). \end{cases}$$

Since the multiple $m_{t_{k-1}}$ must be non negative, we deduce:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1) =$$

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (\exp(Y_{t_k}) - 1) \geq 0] +$$

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (\exp(Y_{t_k}) - 1) < 0]$$

Case 1: $C_{t_{k-1}} > L_{t_{k-1}}$ **.**

Under this assumption, we get:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1) =$$

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[(\exp(Y_{t_k}) - 1) \geq 0] + \mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (\exp(Y_{t_k}) - 1) < 0]$$

Note that we have:

$$m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \Leftrightarrow \exp(Y_{t_k}) > \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1.$$

Therefore:

(1.1) If we have $\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 < 0,$ then the condition $\exp(Y_{t_k}) > \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1$ is satisfied and:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}] = 1 > (1-\epsilon)^{1/T}.$$

(1.2) If the condition $\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 > 0$ is fulfilled, the condition (ConditionY) is equivalent to $Y_{t_k} > \ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1).$ [10]

Since we assume $C_{t_{k-1}} > 0$, $C_{t_{k-1}} > L_{t_{k-1}}$ and $\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 > 0,$ then we deduce:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1] = \mathsf{P}^{\mathsf{G}_{t_{k-1}}}[(\exp(Y_{t_k})-1) \geq 0] +$$

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[0 > Y_{t_k} > \ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1)].$$

Denote by $F^{\mathsf{G}_{k-1}}$ the conditional cdf of Y_{t_k}. We have:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[Y_{t_k} > \ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1)] = 1 - \underbrace{F^{\mathsf{G}_{k-1}}[\ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1)]}.$$

Therefore, the quantile condition

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] \geq (1-\epsilon)^{1/T},$$

is equivalent to:

---

[10]Note that this condition is satisfied as soon as $L_{t_{k-1}} > 0.$

$$1 - F^{\mathsf{G}_{t_{k-1}}}[\ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1)] \ \geq (1-\epsilon)^{1/T},$$

and also to:

$$(F^{\mathsf{G}_{t_{k-1}}})^{-1}([1-(1-\epsilon)^{1/T}]) \geq \ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1).$$

Finally, we have:

1-2-i) If $\exp[(F^{\mathsf{G}_{t_{k-1}}})^{-1} \times [1-(1-\epsilon)^{1/T}]] \ < 1$ (i.e. $Z_{t_{k-1}} < 0$), then $m_{t_{k-1}}$ must satisfy the following constraint:

$$m_{t_{k-1}} \leq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp[(F^{\mathsf{G}_{t_{k-1}}})^{-1}([1-(1-\epsilon)^{1/T}])]] \ - 1}.$$

1-2-ii) If $\exp[(F^{\mathsf{G}_{t_{k-1}}})^{-1} \times [1-(1-\epsilon)^{1/T}]] \ > 1$ (i.e. $Z_{t_{k-1}} > 0$), there is no constraint for $m_{t_{k-1}}$.

Let us examine conditions (1-2-i) and (1-2-ii). They are based on the sign of the term:

$$(F^{\mathsf{G}_{t_{k-1}}})^{-1}([1-(1-\epsilon)^{1/T}]).$$

Lemma *For any real numbers* a *and* b *(* $b > 0$ *) and for any random variable* ϵ *with pdf* f_ϵ *, we have:*

$$\mathsf{P}[a + b\varepsilon \leq x] = \mathsf{P}[\epsilon < \frac{x-a}{b}] = \int_{-\infty}^{\frac{x-a}{b}} f_\epsilon(u)du.$$

Thus the pdf of $a + b\varepsilon$ *is given by:*

$$f(x) = \frac{1}{b} \times f_\varepsilon(\frac{x-a}{b}).$$

We apply this lemma by setting:

$$\begin{cases} a + b\varepsilon = Y_{t_k}, \\ \varepsilon = \varepsilon_{t_k}, \\ a = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}, \\ b = \sigma_{t_k} > 0. \end{cases}$$

We get:

$$f_{Y_{t_k}}^{\mathsf{G}_{k-1}}(y) = \frac{1}{\sigma_{t_k}} \times f_{\varepsilon}\left(\frac{y - (\alpha + \sum_{i=1}^{P} Y_{t_{k-i}})}{\sigma_{t_k}}\right)$$

$$F_{Y_{t_k}}^{\mathsf{G}_{k-1}}(y) = F_{\varepsilon}\left(\frac{y - (\alpha + \sum_{i=1}^{P} Y_{t_{k-i}}}{\sigma_{t_k}}\right),$$

where F_{ε} denotes the common cdf of the random variables ε_{t_k} .

Thus, we deduce:

- The condition in subcase (1-2-i) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_{t_k} \times F_{\varepsilon}^{-1}(1 - (1 - \epsilon)^{1/T}) < 0,$$

- in which case, the conditional multiple must satisfy:

$$m_{t_{k-1}} \leq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp[(\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}) + \sigma_{t_k} \times F_{\varepsilon}^{-1}(1 - (1 - \epsilon)^{1/T})] - 1}.$$

- The condition in subcase (1-2-ii) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_k \times F_{\varepsilon}^{-1}(1 - (1 - \epsilon)^{1/T}) > 0,$$

- with no constraint on the multiple $m_{t_{k-1}}$.

Case 2: $(0 <) C_{t_{k-1}} < L_{t_{k-1}}$ **.**

As previously, the quantile condition is given by:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}] \geq (1 - \epsilon)^{1/T},$$

which is equivalent to:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k}) - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1] \geq (1 - \epsilon)^{1/T}$$

At time t_k, we have two cases:

$$\begin{cases} \exp(Y_{t_k}) < 1 : S_{t_k} < S_{t_{k-1}} \text{ (S decreases)}, \\ \exp(Y_{t_k}) > 1 : S_{t_k} > S_{t_{k-1}} \text{ (S increases)}. \end{cases}$$

Since the multiple $m_{t_{k-1}}$ must be non negative, we have:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1] =$$
$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (\exp(Y_{t_k})-1) \geq 0]$$
$$+ \mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (\exp(Y_{t_k})-1) < 0].$$

Since here $\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (\exp(Y_{t_k})-1) < 0] = 0$, we have:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1] =$$
$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (\exp(Y_{t_k})-1) \geq 0] =$$
$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[m_{t_{k-1}}(\exp(Y_{t_k})-1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1].$$

Consequently, we get the equivalence between the two following conditions:

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}] \geq (1-\epsilon)^{1/T},$$

$$\mathsf{P}^{\mathsf{G}_{t_{k-1}}}[Y_{t_k} \geq \ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1)] \geq (1-\epsilon)^{1/T}.$$

Thus, the quantile condition is also equivalent to:

$$1 - F^{\mathsf{G}t_{k-1}}[\ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1)] \geq (1-\epsilon)^{1/T},$$

$$\ln(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{k-1}} + 1) \leq (F^{\mathsf{G}t_{k-1}})^{-1}[1 - (1-\epsilon)^{1/T}],$$

$$\left(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{m_{t_{k-1}}}+1\right) \leq \exp((F^{G_{t_{k-1}}})^{-1}[1-(1-\epsilon)^{1/T}]).$$

2-i) If $\exp[(F^{G_{t_{k-1}}})^{-1}[1-(1-\epsilon)^{1/T}]] < 1$ (i.e. $Z_{t_{k-1}} < 0$), there is no positive solution for $m_{t_{k-1}}$.

2-ii) If $\exp[(F^{G_{t_{k-1}}})^{-1}[1-(1-\epsilon)^{1/T}]] > 1$ (i.e. $Z_{t_{k-1}} > 0$) then $m_{t_{k-1}}$ must satisfy the following constraint:

$$m_{t_{k-1}} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{\exp[(F^{G_{t_{k-1}}})^{-1}([1-(1-\epsilon)^{1/T}])]] - 1}.$$

Thus, we deduce:

- The condition in subcase (2-i) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_{t_k} \times F_\varepsilon^{-1}(1-(1-\epsilon)^{1/T}) < 0.$$

-
- in which case, the conditional multiple must satisfy:

$$m_{t_{k-1}} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}}-1}{\exp[(\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}) + \sigma_{t_k} \times F_\varepsilon^{-1}(1-(1-\epsilon)^{1/T})] - 1}.$$

-
-
- The condition in subcase (2-ii) is satisfied if and only if

$$\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_{t_k} \times F_\varepsilon^{-1}(1-(1-\epsilon)^{1/T}) > 0;$$

-
-

Appendix 2. Proof of Corollary ES Gaussian.

We have:

$$\Phi_L(m_{t_{k-1}}) =$$

$$L_{t_{k-1}} - \frac{C_{t_{k-1}}[N(l_0(m_{t_{k-1}}))(1-m_{t_{k-1}}) + m_{t_{k-1}} \int_{-\infty}^{l_L(m_{t_{k-1}})} \exp(a_{t_{k-1}} + b_{t_{k-1}}x) \frac{\exp(\frac{-x^2}{2})}{\sqrt{2\pi}} dx}{\int_{-\infty}^{l_L(m_{t_{k-1}})} f_\varepsilon(x) dx}.$$

Therefore:

$$\Phi(m_{t_{k-1}}) = L_{t_{k-1}} - C_{t_{k-1}}[(1 - m_{t_{k-1}}) + \frac{m_{t_{k-1}}}{N(l_L(m_{t_{k-1}}))} \times I_L],$$

with

$$I_L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{l_L(m_{t_{k-1}})} \exp(a_{t_{k-1}} + b_{t_{k-1}} x) \exp(\frac{-x^2}{2}) dx.$$

To calculate the integral I, we use the standard Gaussian transformation:

$$\exp(a + bx) \exp(\frac{-x^2}{2}) = \exp(-\frac{1}{2}[x^2 - 2bx - 2a] =$$

$$\exp(-\frac{1}{2}[(x-b)^2 - b^2 - 2a] = \exp(-\frac{1}{2}(x-b)^2) \exp(\frac{b^2 + 2a}{2}).$$

Then, we deduce:

$$I_L = \frac{1}{\sqrt{2\pi}} \exp(\frac{b^2 + 2a}{2}) \int_{-\infty}^{l_L(m_{t_{k-1}})} \exp -\frac{1}{2}(x-b)^2 dx.$$

Letting $y = x - b$ $(x = y + b)$, we get:

$$I_L = \exp(\frac{b^2 + 2a}{2}) \int_{-\infty}^{l_L(m_{t_{k-1}})+b} \frac{\exp -\frac{1}{2}(y)^2}{\sqrt{2\pi}} dy = \exp(\frac{b^2 + 2a}{2}) \times N[l_L(m_{t_{k-1}}) - b].$$

Consequently, we prove the result of Corollary ES Gaussian.

- Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9, 203-228.
- Awartani, B.M.A., Corradi,V.,2005. Predicting the volatility of the S&P 500 stock index via GARCH models:the role of asymmetries.International Journal of Forecasting 21, 167 -183.
- Balder S., Brandlt, M. ,Mahayni A.,2009. Effectiveness of CPPI strategies under discrete-time trading, Journal of Economic Dynamics and Control, 33, 204-220.
- Bertrand, P., Prigent, J.-L., 2011. Omega performance measure and portfolio insurance. Journal of Banking and Finance 35, 1811-182.
- Bertrand, P., Prigent, J.-L., 2005. Portfolio insurance strategies: OBPI versus CPPI. Finance 26, 5-32.
- Bertrand, P., Prigent, J.-L., 2003. Portfolio insurance strategies: A comparison of standard methods when the volatility of the stock is stochastic. International Journal of Business 8, 461-472.
- Bertrand, P., and Prigent, J-L., 2002. Portfolio insurance: the extreme value approach to the CPPI method. Finance.
- Black, F. and Jones, R., 1987. Simplifying portfolio insurance. The Journal of Portfolio Management, 48-51.
- Black, F. and Perold, A.R.,1992. Theory of constant proportion portfolio insurance. The Journal of Economics, Dynamics and Control, 16, 403-426.
- Black, F.and Rouhani, R.,1989. Constant proportion portfolio insurance and the synthetic put option : a comparison, in Institutional Investor focus on Investment Management, edited by Frank J. Fabozzi. Cambridge, Mass. : Ballinger, pp 695-708.
- Bookstaber, R. and Langsam, J.A., 2000. Portfolio insurance trading rules. The Journal of Futures Markets, 8, 15-31.
- Campbell, J.Y., Viceira, L., 2002. Strategic Asset Allocation: Portfolio Choice for Long-Term Investors. Oxford University Press: Oxford.
- Cesari, R., Cremonini, D., 2003. Benchmarking, portfolio insurance and technical analysis: A Monte Carlo comparison of dynamic strategies of asset allocation. Journal of Economic Dynamics and Control 27, 987-1011.
- Chen, Y. T. and Kuan, C. -M. , 2002. Time Irreversibility and Egarch Effects in US Stock Index Returns, Journal of Applied Econometrics ,17, 565-578.
- Cont R., Tankov P., 2009. Constant proportion portfolio insurance in the presence of jumps in asset prices, Mathematical Finace, 19 , pp. 379--401
- Detemple, J., Garcia, R., Rindishbacher, M., 2003. A Monte Carlo method for optimal portfolios. Journal of Finance 58, 401-446.
- Engle, R., 1982. Autoregressive Conditional Heteroscedasticity with estimates of the variance of U.K. inflation, Econometrica, 50, 987-1008.
- Engel, R.,and Ng, V.,1993. Measuring and testing the impact of news on Volatility ,The Journal of Finance,5,1749 -1778.
- Ested, T. and M. Kritzman,1988. TIP: inPsurance without complexity , Journal of Portfolio Management.
- Föllmer, H., Leukert,P. 1999. Quantile Hedging, Finance and Stochastics, 3, 251-273.

- Goovaerts, M.J., Kaas, S.R., Dhaene, J., 2002. Economical capital allocation derived from risk measures. North American Actuarial Journal 7, 1-16.
- Gourieroux, C., 1997. ARCH Models and Financial Applications. Springer Series in Statistics. Springer: New York, Berlin.
- Gouriéroux C., Monfort A. (1989), Statistique et modèles économétriques: vol n°1, Edition Economica
- Hamidi B., B. Maillet and J.-L. Prigent, 2009b. A Risk Management Approach for Portfolio Insurance Strategies, in Financial Risks, Gouriéroux - Jeanblanc (Eds), Economica, 117-132.
- Hamidi B., E. Jurczenko and B. Maillet, 2009a. A CAViaR Time-Varying Proportion Portfolio Insurance Strategy, Bankers, Markets & Investors 102, 5-22.
- Heynen, R.,Kemna, A., and Vorst, T.,1994. Analysis of the term structure of implied volatilities. Journal Of Financial and Quantitative Analysis,29, 31-56.
- Leland, H.E., Rubinstein, M., 1976. The evolution of portfolio insurance. In: Luskin, D.L. (Ed.), Portfolio Insurance: A Guide to Dynamic Hedging, Wiley
- Markowitz, H., 1952. Porfolio selection. Journal of Finance 7, 77-91.
- Merton, R. C., 1971. Optimum consumption and portfolio rules in a continuous-time models. Journal of Economic Theory 3, 373-413.
- Merton, R. C., 1990. Continuous Time Finance. Basil Blackwell: Cambridge, MA.
- Nelson D.B.,1990. Arch models as diffusion approximations, Journal of Econometrics, 45, 7-39.
- Pedersen, C.S., Satchell, S.E., 1998. An extended family of financial-risk measures. The Geneva Papers on Risk and Insurance Theory 23, 89-117
- Perold, A.R., 1986. Constant Proportion Portfolio Insurance, Harvard Business School, Working paper.
- Perold, A.R. and W. Sharpe , 1988. Dynamic Strategies for Asset Allocations, Financial Analysts Journal, 16-27.
- Poon S. H.,Clive W.J. Granger. 2003. Forecasting Volatility in Financial Markets:A Review,Journal of Economic Literature,Vol.41,N 2.,478-539.
- Prigent, J-L., 2007. Portfolio Optimization and Performance Analysis, Chapman & Hall: Boca Raton, Florida.
- Prigent, J.L., 2001. Assurance du portefeuille: analyse et extension de la méthode du coussin, Bankers, Markets and Investors, 51, 33-39.
- Rockafellar R.T., S. Uryasev, 2002. Conditional value-at-risk for general loss distributions. Journal of Banking and Finance 26, 1443-1471.
- Szegö, G., 2002. Measures of risk. Journal of Banking and Finance 26, 1253-1272.
- Wang, J., Forsyth, P.A., 2011. Continuous-time mean variance asset allocations: A time-consistent strategy. European Journal of Operational Research 209, 184-201.
- Yu, J.-R., Lee, W.-Y, 2011. Portfolio rebalancing model using multiple criteria. European Journal of Operational Research 209, 166-175.
- Yu, B. W.-T., Pang, W.K., Troutt, M. D., Hou, S. H., 2009. Objective comparisons of the optimal portfolios corresponding to different utility functions. European Journal of Operational Research 199, 604-610.
- Zagst, R., Kraus, J., 2011. Stochastic dominance of portfolio insurance strategies: OBPI versus CPPI. Annals of Operations Research 185, 75-103.