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Optimal Portfolio Positioning under Ambiguity

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This paper analyzes the optimality of financial portfolios within utility with ambiguity aversion. It provides a general result about the optimal portfolio profile under ambiguity, in the Anscombe-Aumann framework, using Maccheroni et al. (2006) approach which includes the multiplie priors preferences of Gilboa and Schmeidler (1989) and the multiplier preferences of Hansen and Sergent (2001). Then, the CRRA case is in particular detailed with an ambiguity index based on relative entropy.

Keywords: Portfolio optimization, structured portfolio, ambiguity.

JEL classification: C61, G11, L10..

Introduction

Since the seminal work of von Neumann and Morgenstern (1947), the Expected Utility theory (EU) has been widely applied to model investors attitude towards risk. For the first time, Markowitz (1952) determines the optimal static portfolio solution in the mean-variance framework. This approach is related to quadratic utility functions. Merton (1971) determines the continuous-time optimal portfolio for various utility functions. These fundamental results have been further extended for example by taking account of market incompleteness, of specific constraints on portfolio weights, of labor income and random horizon...as in Cox and Huang (1989), Cvitanic and Karatzas (1996), and, with insurance constraints, in El Karoui *et al.* (2005) and Prigent (2006) (see also Campbell and Viceira, 2002; Prigent, 2007, for a survey about such results). However, some well documented paradoxes, such as the Allais's paradox, have shown that standard utility theory does not model actual attitude towards risk. Allais (1953) suggests that the independence axiom is not often validated from the actual individual behaviors. Cohen and Tallon (2000) mention also that the expected utility theory implies that utility function must simultaneously formalize the choice among alternatives and models risk aversion. Therefore, an investor with a decreasing marginal utility must have necessarily risk aversion.

Among various alternatives[1] to the standard expected utility theory, some authors have suggested that the commun knowledge of the probability distributions is a too strong rational expectation hypothesis. Under this assumption, all individuals have the same opinion about the "true" probability distribution of random events. To overcome the misspecication problem, Hansen and Sargent (2000, 2001) have introduced robust control models.[2] They have argued that uncertainty can be based on ambiguity, which results from the lack of precise information about randomness. Knight (1921) distinguishes between risk and uncertainty. The first one refers to a situation where probabilities are known to guide choice, while uncertainty refers to the situation where information is too vague to define probabilities. The notion of ambiguity has been introduced by Ellsberg (1961). Ellsberg (1961) calls into question Savage's theory (1954), whereby the individual subjective beliefs on the likelihood of the possible states are subjective probabilities. Ellsberg (1961) realizes a simple experience : one urn contains 50 red and 50 black balls and a second one contains a combination of the two but we do not know with which proportions. It is observed that people typically prefer to bet on a ball from the urn with the known mixture than from an urn with unknown proportions. This shows that individual choice under uncertainty depends on the consequences, on probabilities associated with these consequences and also on the confidence that individuals accord to these probabilities. This evidence reflects aversion to the ambiguity: individuals prefer to act on known rather than unknown probabilities. Gilboa and

[1] See for instance the weighted utility introduced in Chew (1989), the rank-dependent utility (see Segal, 1989) or the cumulative prospect theory of Kahneman and Tversky (1992).

[2] The robust preferences approach considers that the individual objective functions take account of the possibility that the model used by the individual may be false and only be an approximation of the true model.

Schmeidler (1989) have considered the so-called "maxmin expected utility preferences", which assumes the existence of multiple priors.[3] Maccheroni *et al.* (2006) propose a model in accordance to the standard Anscombe and Aumann (1963) approach, based on specific assumptions on both the utility function and the ambiguity index. This model includes the case of multiple priors preferences considered by Gilboa and Schmeidler (1989), the multiplier preferences introduced by Hansen and Sargent (2001), and also the mean-variance preferences of Markowitz (1952) and Tobin (1958). Asano (2011) focus on the portfolio inertia in the context of Knightian uncertainty. He considers two cases: the first one corresponds to preference represented by the Choquet expected utility theory and the second one to preference represented by the maxmin expected utility axiomatized by Gilboa and Schmeidler (1989). He analyzes the effect of the uncertainty on the spread between buying and selling prices in stock markets. Qu (2011) proposes a generalization of the maxmin expected utility model and of the subjective expected utility model. In this framework, ambiguity and unambiguity are distinguished through the belief representation. Taking the Knightian distinction into account, Qu (2011) suggests a subjective definition of ambiguity, especially in the context of biseparable preference.

Portfolio optimization under ambiguity has been examined in various frameworks. For the standard portfolio allocation as introduced by Markowitz (1952), the investor must choose his portfolio weights at the initial date. Portfolio returns are linear combinations of asset returns. In that case, Pflug and Wozabal (2007) use a maximin criterion based on a confidence set for the probability distribution. They illustrate the tradeoff between return, risk and robustness with respect to ambiguity and provide a monetary value of the information (see also Wozabal (2012) for the case of non-parametric ambiguity sets). Calafiore (2007) determines the optimal robust portfolios when assuming that a nominal discrete return distribution is given, while the true distribution is unknown except that it lies within a given distance from the nominal one computed according to the Kullback-Leibler divergence criterion. He determines portfolios that minimize the maximum among all the allowable distributions of a given weighted risk-mean objective (in particular, the standard variance and absolute deviation measures). Additionally, Pflug *et al.* (2012) show that the uniform investment[4] strategy is rational for investors facing a significant high degree of ambiguity about loss distributions, for a large class of risk measures. Koziol *et al.* (2011) deal with ambiguity of institutional investors towards specific assets. By estimating the average portfolio weightings for standard and alternative asset classes of 119 institutional investors, the model can be calibrated to identify the ambiguity factors of each asset type. They show that institutional investors are strongly ambiguity-averse and that equities and bonds have much lower ambiguity than other investments such as real estate investments, private equities, and hedge funds. In the continous-time framework, Fei (2007) examines the optimal portfolio choice with respect to the recursive multiple-priors utility. He provides explicitly the optimal consumption and portfolio values for power and logarithmic felicity functions. Liu (2011)

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[3]For other models of ambiguity with multiple priors, see also Epstein and Wang (1994) and Chen and Epstein (2002).

[4]The equally weighting 1/N investment strategy.

examines also the same problem where expected returns of a risky asset follow a hidden Markov chain. He proves that ambiguity aversion emphasizes the importance of hedging demands in the optimal portfolio strategies.

In this paper, we provide the optimal portfolio payoff within ambiguity, in the optimal portfolio positioning framework introduced by Leland (1980), and Brennan and Solanki (1981). In this framework, the portfolio value is a function of a given benchmark.[5] The portfolio payoff maximizes the investor's expected utility while taking account of the ambiguity index. The investor's risk aversion and his ambiguity index characterize the optimal portfolio profile, which involves option-based strategies. [6] A particular case of optimal positioning is the portfolio insurance theory, introduced by Leland and Rubinstein (1976). This theory usually considers portfolio payoffs which are functions of a benchmark (a specified portfolio of common assets).[7] At maturity, downside risk is limited (the investor can receive a given percentage of his initial capital, even in bearish markets), while the investor can participate in upside markets. However, more specific insurance constraints can be introduced, for example for institutional investors (see e.g. Bertrand *et al.* (2001) for quite general insurance constraints).[8].

The paper is organized as follows. Section 2 presents a survey about ambiguity theory, mainly the Maccheroni *et al.* (2006) approach. Section 3 provides the general result about portfolio positioning under ambiguity aversion. Within this framework, Section 4 illustrates the general result by examining a fundamental example that emphasizes the role of both aversions to risk and ambiguity.

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[5]Note that portfolio positioning refers to static strategies. But actual portfolio hedging strategies correspond indeed to discrete-time trading. Additionnally, structured portfolio management is based in particular on initial positioning on financial derivatives.

[6]Carr and Madan (2001) prove that the existence of out-of-the-money European puts and calls of all strikes allows the determination of the optimal positioning in a complete market. This hypothesis is justified when there is a large number of option strikes (e.g. the S&P500, for example).

[7]The *Option Based Portfolio Insurance* (OBPI), has been introduced by Leland and Rubinstein (1976). It consists of a portfolio invested in a risky asset S (usually a financial index such as the *S&P*) covered by a listed put written on it. Whatever the value of S at given horizon T, the portfolio value is always above the strike K of the put.

[8]See also El Karoui *et al.* (2005) who determine the optimal portfolio with an American capital guarantee.

The concept of ambiguity

Two main notions of ambiguity have been proposed. Epstein (1999) considers ambiguity neutrality with respect to probabilistically sophisticated preferences. Ghirardato and Marinacci (2002) identify ambiguity neutrality with subjective expected utility preferences. They consider subjective expected utility preferences as ambiguity neutral preferences. The notion of ambiguity aversion is what provides a foundation for the standard comparative statics exercises in ambiguity for multiple priors preferences that are based on the size of the set of priors. This is the case of Hansen and Sargent (2001) multiplier preferences, which are easily seen to be probabilistically sophisticated. Consider an individual that has to make choices and faces limited information about what may happen. Usually, this individual will be cautious.[9]. Remaining in the Von Neumann-Morgenstern formalization and particularly in the Savage's model, Gilboa and Schmeidler (1989) propose to accomodate ambiguity in economic decision making and assume that in the presence of ambiguity the individual cannot identify a single probability distribution over states of nature. Thus, he considers multiple probability distributions and then evaluates his choices according to the worst probability distribution for that choice. This is the approach based on multiple priors. The decision model is called the "maxmin expected utility" (MEU). This model is flexible,allow a distinction between risk and ambiguity and can captures the preference of ambigity aversion. Maccheroni *et al.* (2006) characterize the preferences under ambiguity by introducing both an utility function U on outcomes and an ambiguity index C on the set of probabilities defined on the random events. Thus, they consider the following representation of preferences:

For all random variables X and Y which represent results or consequences and with values in $[-M,M]$, we have:

$$X \succeq Y \Leftrightarrow$$

$$Min_{\mathsf{P}\in\Delta} \int U(X)d\mathsf{P} + C(\mathsf{P}) \geq Min_{\mathsf{P}\in\Delta} \int U(Y)d\mathsf{P} + C(\mathsf{P}).$$

The function U corresponds to decision risk attitude. Index C represents the individual attitude towards ambiguity. This representation of preferences includes both the multiple priors preferences of Gilboa and Schmeidler (1989) and the multiplier preferences of Hansen and Sargent (2000, 2001). The MEU criterion of Gilboa and Schmeidler (1989) corresponds to the case where $C = 0$ and Δ is a convex set. This set is viewed as the set of priors of the individual. Ambiguity is associated to the multiplicity of priors. Hansen and Sargent (2000, 2001) propose a robust preference approach where the individual is uncertain about his modelisation of random events. The ranking of decisions is based on:

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[9]This attitude can be relevant in many cases including Ellsberg-type choice situations

$$X \succeq Y \Leftrightarrow$$
$$Min_{\mathrm{P}\in\Delta} \int U(X) d\mathrm{P} + \theta R(\mathrm{P},\mathrm{Q}) \geq Min_{\mathrm{P}\in\Delta} \int U(Y) d\mathrm{P} + \xi R(\mathrm{P},\mathrm{Q}),$$

where $R(\mathrm{P},\mathrm{Q})$ denotes the relative entropy with respect to the given probability distribution Q. This kind of preferences is known as "multiplier preferences". The parameter ξ represents the weight that individual gives to the possibility that Q is not the appropriate probability distribution, due to the lack of information. Maccheroni *et al.* (2006) establish theoretical connections between the two approaches. For a given set Δ, the lower is C, the higher is the ambiguity aversion. Note that the mean-variance preference of Markowitz (1952) and Tobin (1958) corresponds to $U(X) = X$ and $C(X) = -\frac{\varphi}{2} Var(X)$. This theoretical result is an important step in the modeling of decision theory. It allows the evaluation of risky portfolios in an environment characterized by uncertainty with a rational procedure.

Optimal positioning under ambiguity

This section extends previous results about expected utility maximization of Leland (1980), Brennan and Solanki (1981), Carr and Madan (2001) and Prigent (2006) to the expected utility with ambiguity. Suppose that the investor maximizes an expectation of his utility U in the presence of ambiguity. He is a price taker (for example, his benchmark S is the SP&500 and his investment is too weak to modify the index value). This attitude corresponds to an investor whose portfolio value is a function of only the terminal value of the risky asset.

Introductory example

We consider first a particular financial portfolio with three assets: the riskless asset B, the risky asset S and a put option written on S. We suppose that the risky asset S follows a geometric Brownian motion. The Put is evaluated within the Black-Scholes model. Suppose that the interest rate r is constant and that the stock price has the following Lognormal distribution defined by:

$$S_T = S_0 exp\left[dT + \sigma \sqrt{T}\, X \right],$$

where the distribution of X is the standard Gaussian $\mathrm{N}(0,1)$.[10] We denote by $P_0(K)$ the initial Put value with strike K. At maturity T, the portfolio value with the three respective asset shares α, β and γ is given by:

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[10]It corresponds for example to the distribution of a geometric Brownian motion $(S_t)_t$ in a continuous-time framework, satisfying

$$S_t = S_0 exp[(\mu - 1/2\sigma^2)t + \sigma W_t],$$

with $d = (\mu - 1/2\sigma^2)$ and where W denotes the standard Brownian motion.

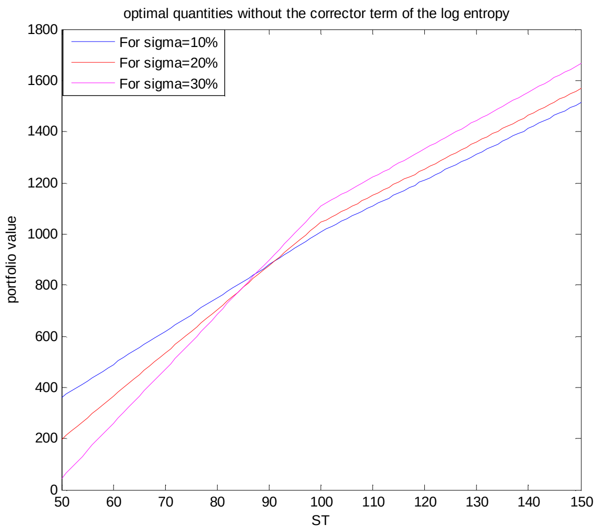
$$V_T = \alpha B_T + \beta S_T + \frac{V_0 - \alpha B_0 - \beta S_0}{P_0(K)}(K - S_T)^+.$$

Portfolio value with two assets, at maturity T is given by:

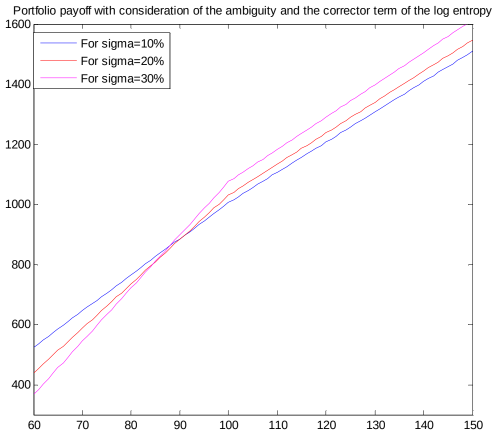
$$V_T = \alpha B_T + \beta S_T.$$

To illustrate numerically the optimal portfolio, we consider the following base parameters values: $B_0 = 1, S_0 = 100, V_0 = 1000, r = 2\%$. We assume that ambiguity only concerns the drift term μ . We consider a HARA utility defined by $U(V) = \frac{(V-V^*)^{1-\phi}}{1-\phi}$ with $\phi \neq 1$. The index C corresponds to the LogEntropy criterion with respect to the probability P_{μ_0} associated to the drift value μ_0 . Probabilities P_μ correspond to drifts μ . We denote by f_{μ,S_T} the probability distribution function (pdf) of asset S_T with respect to P_μ and by g_{μ,μ_0} the ratio $f_{\mu,S_T}/f_{\mu_0,S_T}$. We have to solve $\max_{\alpha,\beta \text{ and } \gamma} \min_{\mu \in [\underline{\mu},\bar{\mu}]} \mathsf{E}_{\mathsf{P}_\mu}[U(V_T)] + C(\mathsf{P}_\mu)$, with $\mathsf{E}_{\mathsf{P}_\mu}[U(V_T)] = \int_0^{+\infty} U\Big[\alpha B_0 e^{rT} + \beta S_0 e^s + \frac{V_0-\alpha B_0-\beta S_0}{P_0(K)}(K - S_0 e^s)^+\Big] f_{\mu_0,S_T}(s) ds,$
and $C(\mathsf{P}_\mu) = \theta \int_0^{+\infty} g_{\mu,\mu_0}(s)[\log(g_{\mu,\mu_0})(s)] f_{\mu_0,S_T}(s) ds$.

REMPLACER PAR UNE SURFACE $h^*(\sigma, S_T)$



In the following graph, we illustrate portfolio payoff for the case with ambiguity and with the corrector term, for several value of sigma, when all other parameters remain unchanged. We observe that if we increase the value of the volatility, the negative value of γ optimal quantities invested on the put increase.



If we take account of the corrector term in the optimization problem, the quantities invested on the put option decrease.

The optimal portfolio profile

We search now for the general optimal portfolio profile as function of the risky benchmark S_T. Assume the existence of three basic types of financial assets: the cash associated to a discount factor N, the bond B and the system of stock prices S (one financial index for example). We suppose that the investor determines an optimal payoff h which is a function defined on all possible values of the assets (N_T, B_T, S_T) at maturity T. If the market is complete, this payoff can be achieved by the investor. The market can be complete for example if the financial market evolves in continuous-time and all options can be dynamically duplicated by a perfect hedging strategy. It can be still complete if for example, in a one-period setting, European options of all strikes are available on the financial market. In this setting, the inability to trade continuously potentially induces investment in cash, asset B, asset S and all European options with underlying assets B and S (if cash and bond are non stochastic, only European options on S are required). The market can be also incomplete. In that case, the solution given in this section is only theoretical but still interesting to know since the optimal payoff can be approximated by investing on traded assets (in practice, the investor defines an approximation method, which may take transaction costs or liquidity problems into account). Under the standard condition of no-arbitrage, the assets prices are calculated under risk neutral probabilities. If markets exist for out-of-the-money European puts and calls of all strikes, then it implies the existence of an unique risk-neutral probability that may be identified from option prices (see Breeden and Litzenberger, 1978). Otherwise, if there is no continuous trading, generally the market is incomplete and a one particular risk-neutral probability Q_r is used to price the options. It is also possible that stock prices change continuously but the market may be still dynamically incomplete. Again, it is assumed that one risk-neutral probability is selected.

Assume that prices are determined under such measure Q_r. Denote by $\frac{d\mathsf{Q}_r}{d\mathsf{P}}$ the Radon-Nikodym derivative of Q_r with respect to the historical probability P. Denote by N_T the discount factor and by M_T the product $N_T \frac{d\mathsf{Q}_r}{d\mathsf{P}}$. Due to the no-arbitrage condition, the budget constraint corresponds to the following relation:

$$V_0 = \mathsf{E}_{\mathsf{Q}_r}[h(N_T, B_T, S_T) N_T] = \mathsf{E}_{\mathsf{P}}[h(N_T, B_T, S_T) M_T].$$

In what follows, the utility U of the investor is supposed to be increasing and piecewise differentiable. For the optimal positioning, the portfolio value V is a function of the basic assets: $V = h(N_T, B_T, S_T)$. Therefore, the investor has to solve the following maximization problem:

$$Max_h \; Min_{\mathsf{P} \in \Delta} (\mathsf{E}_{\mathsf{P}}[U(h(N_T, B_T, S_T))] + C[\mathsf{P}]).$$

under

$$V_0 = \mathsf{E}_{\mathsf{P}}[h(N_T, B_T, S_T) M_T].$$

Assume that $X_T = (N_T, B_T, S_T)$ has a pdf denoted by f_{X_T}. Then, the expectation $\mathsf{E}_{\mathsf{P}}[U(h(N_T, B_T, S_T)]$ is equal to

$$\int U[h(x)] f_{X_T} dx.$$

To simplify the presentation of main results, we suppose as usual that function h fulfils:

$$\int_{\mathsf{R}^{+3}} h^2(x) f_{X_T} dx < \infty, \text{ for all } f_{X_T} \in F,$$

where F denotes the set of pdf corresponding to the set of probability distributions Δ.

It means that $h \in \mathsf{H} = \mathsf{L}^2(\mathsf{R}^{+3}, \mathsf{P}_{X_T}(dx), \mathsf{P}_{X_T} \in \Delta),$ which is the set of the measurable functions with squares that are integrable on R^{+3} with respect to all the distributions $\mathsf{P}_{X_T}(dx)$ belonging to the set Δ. Under mild assumption about the payoffs and the set Δ of multipriors, we can deduce a first general result.

Assumption 1: The utility function U *is strictly concave and continuous;*
Assumption 2: For any $h \in \mathsf{H},$ *the functional* $\mathsf{E}_{\mathsf{P}}[U(h(N_T, B_T, S_T))] + C[\mathsf{P}]$ *is continuous and quasiconvex on* $\Delta.$
Assumption 3: We search the solution in the subset of continuous functions belonging to H *;*
Assumption 4: The set Δ *of multipriors is compact.*

Proposition *Under Assumptions (1,2,3,4), the optimal payoff* h^* *exists and corresponds to the optimal solution for a given* $\mathsf{P}^*_{X_T} \in \Delta$.

Proof By Assumptions (1,2,3), the functional $(\mathsf{E}_{\mathsf{P}}[U(h(N_T, B_T, S_T))] + C[\mathsf{P}])$ is a real function which is continuous with respect to $\mathsf{P} \in \Delta$.
Therefore, if Δ is compact (Assumption 4), for any given payoff h, the solution of $Min_{\mathsf{P} \in \Delta}(\mathsf{E}_{\mathsf{P}}[U(h(N_T, B_T, S_T))] + C[\mathsf{P}])$ is reached at a probability $\mathsf{P}(h) \in \Delta.$
Using MinimaxTheorem [11], we deduce the result.

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[11]**Minimax Theorem (Sion, 1958): (Saddle-point)** Let C and K be two closed convex sets in two topological vector spaces X and Y respectively. Let further $F(x,y)$: $C \times K \to R$ be a function which is quasiconcave in x and quasiconvex in y. If F is upper (or lower) semi continuous in x and lower semi continuous in y, while K is compact then the function $F(x,y)$ possesses a saddle-value on $C \times K$ and

$$\sup_{x \in C} \inf_{y \in K} F(x,y) = \inf_{y \in D} \sup_{x \in K} F(x,y).$$

Corollary *Sufficient conditions to guarantee the quasiconvexity of the functional* $\mathsf{E}_{\mathsf{P}}[U(h(N_T,B_T,S_T))] + C[\mathsf{P}]$ *with respect to* $\mathsf{P} \in \Delta$.

- *First, we assume that the functional* $C[\mathsf{P}]$ *itself is continous and quasiconvex on* Δ.
- *Second, since* $\mathsf{E}_{\mathsf{P}}[U(h(N_T,B_T,S_T))]$ *is convex with respect to* P *(by linearity), then the quasiconvexity of* $C[.]$ *implies the quasiconvexity of the following sum* $\mathsf{E}_{\mathsf{P}}[U(h(N_T,B_T,S_T))] + C[\mathsf{P}]$ *(see Debreu and Koopmans, 1982; Crouzeix and Lindberg, 1986). Such property is satisfied in next basic example where the ambiguity is about the true values of* μ *and* σ *.*

In what follows, we determine first-order conditions:

Consider a new functional Γ_{U_f} associated to the utility function U . It is defined on the space by: For any $Y \in \mathsf{H}$,

$$\Gamma_{U,C}(Y) = Min_{\mathsf{P}\in\Delta}(\mathsf{E}_{\mathsf{P}}[U(Y)] + C[\mathsf{P}]).$$

$\Gamma_{U,C}$ is usually called a Nemitski functional associated to U when U is concave and $C = 0$ (see Ekeland and Turnbull (1983) for definition and basic properties).

Denote also by g the density of M_T with respect to P .[12] Assume that g is a function defined on the set of the values of X_T and $g \in \mathsf{H}$. Then, the optimization problem is reduced to:

$$Max_{h\in\mathsf{H}} Min_{f_{X_T}\in\widetilde{\Delta}}\left(\int_{\mathsf{R}^{+3}} (U[h(x)]).f_{X_T}(x)dx + C[f_{X_T}]\right),$$

$$\text{under } V_0 = \int_{\mathsf{R}^{+3}} h(x)g(x)f_{X_T}(x)dx.$$

Denote by $\widetilde{\Delta}$ the set of pdf f_{X_T} coresponding to $\mathsf{P}_{X_T} \in \Delta$, and by $\widehat{\Gamma}_{U,C}$ the functional defined on the set H by:

$$\widehat{\Gamma}_{U,C}(h) = Min_{f_{X_T}\in\widetilde{\Delta}}\left(\int_{\mathsf{R}^{+3}} (U[h(x)]).f_{X_T}(x)dx + C[f_{X_T}]\right).$$

Let Λ be the linear functional such that:

$$\Lambda(h) = \int_{\mathsf{R}^{+3}} h(x)g(x)f_{X_T}(x)dx.$$

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[12]Function g is the Radon-Nikodym density $d\mathsf{Q}/d\mathsf{P}$.

Proposition *Assume that* $\widehat{\Gamma}_{U_{f,h_0}}(.)$ *is differentiable. Then, every relative maximum* h^* *of* $\widehat{\Gamma}_{U_{f,h_0}}$ *under condition (budget) must necessarily satisfy the first-order condition: there exists a scalar* λ *such that*

$$\frac{\partial \widehat{\Gamma}_{U,C}(h)}{\partial h} = \lambda \Lambda.$$

Proof This property is a consequence of general results about optimization under constraints, when both the functional to optimize and the function characterizing the constraint are differentiable. Here, we note that the derivative of the continuous linear functional L is equal to itself. Thus, we deduce the result.

Proposition *The optimal portfolio payoff* h^* *of* $\widehat{\Gamma}_{U_{f,h_0}}$ *under condition (budget) is characterized by the first-order condition: there exists a scalar* λ *such that:*

$$\frac{\partial \widehat{\Gamma}_{U,C}(h)}{\partial h} = \lambda g(.).$$

Furthermore, if the function $\Theta_{U,C}(z) = \frac{\partial \widehat{\Gamma}_{U,C}(.)}{\partial h}$ *is invertible for all* x *, then the optimal payoff* h^* *is given by:*

$$h^* = \Theta_{U,C}^{-1}(\lambda g),$$

where λ *is the scalar Lagrange multiplier such that:*

$$V_0 = \int_{\mathrm{R}^{+3}} \Theta_{U,C}^{-1}(\lambda g(x)) g(x) f_{X_T}(x) dx.$$

Properties of the optimal portfolio

Suppose for example that cash and bond are non stochastic. Then, the properties of the optimal payoff h^* as function of the benchmark S can be analyzed.

Corollary *If function* $\Theta_{U,C}$ *is decreasing,* h^* *is an increasing function of the benchmark* S_T *if and only if the density* g *is a decreasing function of* S_T *. If function* $\Theta_{U,C}$ *is increasing, it is the converse.*

Proposition *Assume that functional* $\Theta_{U,C}$ *has an inverse* $\Theta_{U,C}^{-1}$ *. The optimal payoff* h^* *must satisfy:*

$$h^{*\prime}(s) = \left(-\frac{\Theta'_{U,C}(h(s))}{\Theta''_{U,C}(h(s))}\right) \times \left(-\frac{g'(s)}{g(s)}\right).$$

Proof Using the relation $\lambda = \Theta'_{U,C}/g$, standard differential calculus leads to:

$$h'(s) = \left(-\frac{\Theta'_{U,C}(h(s))}{\Theta''_{U,C}(h(s))}\right) \times \left(-\frac{g'(s)}{g(s)}\right).$$

Introduce the function $T_o(h(s))$ defined by:

$$T_o(h(s)) = -\frac{\Theta'_{U,C}(h(s))}{\Theta''_{U,C}(h(s))}.$$

Note that, if U is always concave and there is no regret/rejoice (standard case), the function $T_o(h(s))$ is called the tolerance of risk and corresponds to the inverse of the absolute risk-aversion. As it can be seen, $h'(s)$ depends on $T_o(h(s))$. The design of the optimal payoff can also be specified. Denote

$$Y(s) = -\frac{g'(s)}{g(s)}.$$

Differentiating twice with respect to $s,$ we get:

Corollary *The second-order derivative of payoff* h *is given by:*

$$h''(s) = [T'_o(h(s)) + \frac{Y'(s)}{Y(s)^2}] \times [T_o(h(s))Y^2(s)].$$

From above relation, conditions of concavity/convexity can be deduced.

Hedging of the optimal portfolio

As proved in Carr and Madan (1997), it is possible to explicitly identify the investment strategy that must be taken in order to achieve a given payoff h that is twice differentiable. Suppose for example that the interest rate is non stochastic. The portfolio $h(S)$ is duplicated by an unique initial position of $h(S_0) - h'(S_0)S_0$ unit discount bonds, $h'(S_0)$ shares and $h(K)dK$ out-of-the-money options of all strikes K :

$$h(S) = [h(S_0) - h'(S_0)S_0] + h'(S_0)S + \int_0^{S_0} h''(K)(K - S)^+ dK + \int_{S_0}^{\infty} h''(K)(S - K)^+ dK.$$

Generally, h_0 is increasing and h^*. Therefore, the optimal payoff is an increasing function of the benchmark. If h is not differentiable, it is approximated by a sequence of twice differentiable payoff functions h_n. Then, since the payoff h_n are twice differentiable, h_n are duplicated by initial positions of $h_n(S_0) - h'_n(S_0)S_0$ unit discount bonds, $h'_n(S_0)$ shares and $h_n(K)dK$ out-of-the-money options of all strikes K :

$$h_n(S) = [h_n(S_0) - h'_n(S_0)S_0] + h'_n(S_0)S \\ + \int_0^{S_0} h''_n(K)(K-S)^+ dK + \int_{S_0}^{\infty} h''_n(K)(S-K)^+ dK.$$

Fundamental example

In what follows, previous theoretical results are illustrated for the standard expected utility and the utility with aversion to ambiguity. Option prices are assumed to be determined in the well-known Black and Scholes framework.[13]

The financial market

In what follows, we study an optimization problem with ambiguity corresponding to multiple priors that an investor can have on the instantaneous market return μ. We analyze two cases: for the first one, we do not take the corrector term of the log entropy into account; for the second one, we take it into account.

Suppose that the interest rate r is constant and the stock price has a Lognormal distribution given by:

$$S_T = S_0 exp\left[mT + \sigma \sqrt{T}\ X \right],$$

where the distribution of X is the standard Gaussian $\mathsf{N}(0,1)$. For example, in a continuous-time framework, consider a geometric Brownian motion $(S_t)_t$ given by:

$$S_t = S_0 exp[(\mu - 1/2\sigma^2)t + \sigma W_t],$$

with $m = (\mu - 1/2\sigma^2)$.

The probability density function (pdf) f of S_T is given by:

$$f(s) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left(-\frac{1}{2\sigma^2 T}\left[\ln\left(\frac{s}{S_0} \right) - mT \right]^2 \right) \mathsf{I}_{s>0}.$$

The cumulative distribution function (cdf) F of S_T is given by: (N denotes the cdf of the

[13]This case is examined since it is the most used in practice. Other cases can also be considered if the Log return of the risky asset is no longer Gaussian.

standard normal distribution $\mathrm{N}(0,1)$)

$$F_S(s) = N\left[\frac{\ln\left(\frac{s}{S_0}\right) - mT}{\sigma\sqrt{T}}\right].$$

Introduce the following notations:

$$\theta = \frac{\mu - r}{\sigma} \text{ (Sharpe ratio)}, \; A = -\frac{1}{2}\theta^2 T + \frac{\theta}{\sigma} mT, \; \chi = e^A (S_0)^{\frac{\theta}{\sigma}}, \; \kappa = \frac{\theta}{\sigma}.$$

Recall that in the Black and Scholes model, since we have:

$$W_T = \frac{\ln\left(\frac{S_T}{S0}\right) - mT}{\sigma},$$

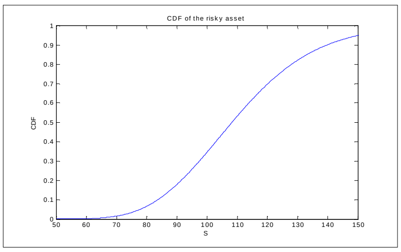
we deduce that the conditional expectation g of $\frac{d\mathrm{Q}}{d\mathrm{P}}$ under the σ -algebra generated by S_T is given by:

$$g(S_T) \text{ with } g(s) = \chi s^{-\kappa}.$$

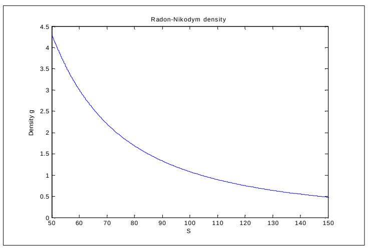
We apply the previous general results to solve the optimization problem. The solutions are illustrated for the following numerical values of financial market parameters:

$$r = 2\%, \; \sigma = 20\%, B_0 = 1, S_0 = 100.$$

The initial investment is $V_0 = 1000$ and the time horizon T is equal to 1 (one year).



Cdf of the risky asset S_T



Radon-Nikodym density $d\mathrm{Q}/d\mathrm{P}$

Standard expected utility

Assume that the utility function of the investor is a CRRA utility:

$$U(v) = \frac{v^{1-\phi}}{1-\phi},$$

with $\phi \neq 1$ from which we deduce $J(y) = y^{-\frac{1}{\phi}}$.

Proposition *The optimal payoff is given by:*

$$h_{EU}^{*}(s) = \frac{V_0 e^{rT}}{\int_0^{\infty} g(s)^{\frac{1-\phi}{\phi}} f(s) ds} \times g(s)^{-\frac{1}{\phi}}.$$

Therefore, $h_{EU}^{*}(s)$ *satisfies:*

$$h_{EU}^{*}(s) = d \times s^{m} \text{ with } d = c\chi^{-\frac{1}{\phi}}, \text{ and } m = \frac{\kappa}{\phi} > 0.$$

Remark *Note that* h^{*} *is increasing. This property is satisfied for all concave utilities, as soon as the density* g *is decreasing, for instance within the Black-Scholes asset pricing framework.*

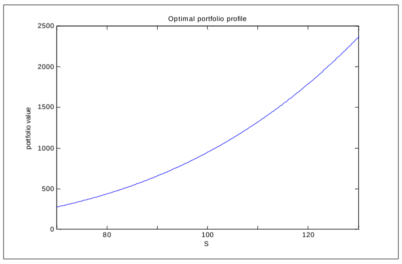
Corollary *The concavity/convexity of the optimal payoff is determined by the comparison between the relative risk-aversion* γ *and the ratio* $\kappa = \frac{\mu - r}{\sigma^2}$ *, which is the Sharpe ratio divided by the volatility* σ *:*[14]

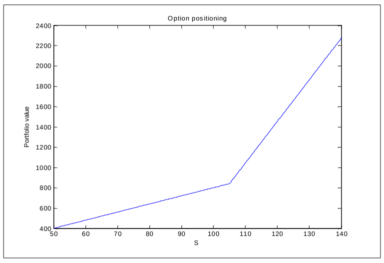
i) h_{EU}^{*} *is concave if* $\kappa < \phi$.

ii) h_{EU}^{*} *is linear if* $\kappa = \phi$.

iii) h_{EU}^{*} *is convex if* $\kappa > \phi$.

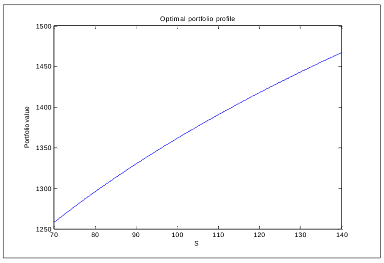
[14] See e.g. Prigent (2006, 2007).

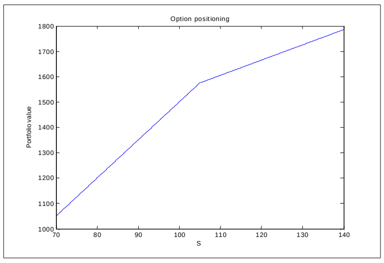




Portfolio value for $\phi = 1/2$ Corresponding buy/sell options

According to the financial values and the relative risk aversion, we can get a convex payoff (for example, for $\mu = 7\%$). The (approximated) corresponding position on option markets is for example $V_T = -3400B_T + 40S_T + 32.7(K - S_T)^+$ with strike $K = 105$.





Portfolio value for $\phi = 1/2$ Corresponding buy/sell options

For $\mu = 3\%$, the (approximated) corresponding position on option markets is for example $V_T = 1000B_T + 5.7S_T + 3.4(S_T - K)^+$ with strike $K = 105$.

Utility with ambiguity

Assume that the utility function is a power function $U(x) = \frac{x^{1-\phi}}{1-\phi}$ (CRRA case) with relative risk aversion $\phi > 0$ and $\phi \neq 1$. The set of multiple priors Δ correspond to ambiguity with respect to parameters μ and σ . Under $\mathsf{P}_{\mu,\sigma}$, the risky asset price is given by:

$$S_t = S_0 exp[(\mu - 1/2\sigma^2)t + \sigma W_{\mu,t}],$$

where $W_{\mu,\sigma,t}$ has a standard Gaussian distribution with respect to $\mathsf{P}_{\mu,\sigma}$. The compact set Δ corresponds to a compact set of pairs (μ, σ) .

Case 1. Consider first the case $C = 0.$ It corresponds to the criterion maxmin expected utility of Gilboa and Schmeidler (1989).
We have to solve:

$$Max_{h \in \mathsf{H}}\, Min_{\mathsf{P} \in \Delta}(\mathsf{E}_{\mathsf{P}}[U(h(S_T))]).$$

From Propositions (MinimaxTheorem) and (CRRA Profile), we deduce that the optimal solution for the CRRA case is given by:

$$h^*(S_T, Y) = d.S_T^{\frac{\kappa}{\phi}},$$

for a particular value of the pair (μ, σ) belonging to Δ. Recall that, for any power p, we have $E_{\mathsf{P}\mu}[S_T^p] = S_0^p \exp[p(\mu + 1/2(p-1)\sigma^2)T]$.
Applying budget constraint, the coefficient d is equal to

$$d = \frac{V_0 e^{rT}}{E_{\mathsf{Q}_r}\left[S_T^{\frac{\kappa}{\phi}}\right]} = \frac{V_0 e^{rT}}{S_0^{\frac{\kappa}{\phi}} \exp\left[\frac{\kappa}{\phi}(r + 1/2(\frac{\kappa}{\phi} - 1)\sigma^2)T\right]}.$$

Thus, the expected utility is given by

$$\frac{1}{1-\phi} E\left[\left(dS_T^{\frac{\kappa}{\phi}}\right)^{1-\phi}\right] = \frac{(V_0 e^{rT})^{1-\phi}}{1-\phi} \frac{\exp\left[\frac{\kappa}{\phi}(1-\phi)(\mu + 1/2(\frac{\kappa}{\phi}(1-\phi) - 1)\sigma^2)T\right]}{\exp\left[\frac{\kappa}{\phi}(1-\phi)(r + 1/2(\frac{\kappa}{\phi} - 1)\sigma^2)T\right]}.$$

Therefore, by simplifying previous expression, we deduce that the optimal pair (μ^*, σ^*) corresponds to the maximization of $\frac{(V_0 e^{rT})^{1-\phi}}{1-\phi} E\left[\exp\left(\frac{1}{2}\theta^2 T \frac{1-\phi}{\phi}\right)\right]$.

Proposition *In the CRRA case and for the Geometric Brownian motion, the optimal solution with ambiguity aversion corresponds to the optimal solution for the standard expected utility criterion when (* μ *,* σ *) minimizes the absolute value of the Sharpe ratio* $\theta = \frac{\mu - r}{\sigma}$.

Corollary *Assume that ambiguity is only about* $\mu \in [\underline{\mu}, \overline{\mu}]$ *(the volatility* σ *is supposed to be fixed). Then,if* $\underline{\mu} \geq r$ *, the optimal solution of Problem (ExampleGS) corresponds to an optimal solution of the standard expected utility maximization for the lowest value* $\underline{\mu}$ *of parameter* μ*. The optimal payoff with ambiguity* $h^*_{U,C}$ *is a positive power function of the risky asset. It is gievn by:*

$$h^*_{U,C}(s) = d(\underline{\mu}) \times s^{m(\underline{\mu})} \text{ with } m(\underline{\mu}) = \frac{\underline{\mu} - r}{\sigma^2} \frac{1}{\phi}.$$

Case 2. Introduce now the Log Entropy multiplied by parameter θ as ambiguity aversion. Let P_{μ_0} be the reference probability distribution.
We must solve the following optimization problem:

$$Max_{h \in \mathsf{H}} Min_{\mathsf{P} \in \Delta} (\mathsf{E}_{\mathsf{P}}[U(h(S_T))] + C[\mathsf{P}]),$$

with

$$C[\mathsf{P}] = \mathsf{E}_{\mathsf{P}_{\mu_0}}\left[\left(\frac{d\mathsf{P}_\mu}{d\mathsf{P}_{\mu_0}}\right)Log\left(\frac{d\mathsf{P}_\mu}{d\mathsf{P}_{\mu_0}}\right)\right],$$

In that framework, we can determine explicitly the LogEntropy for a given reference P_{μ_0}.

Lemma *For Lognormal distributions of the risky asset, the Log Entropy of probability distribution* P_μ *with respect to* P_{μ_0} *is given by:*

$$\mathsf{E}_{\mathsf{P}_{\mu_0}}\left[\left(\frac{d\mathsf{P}_\mu}{d\mathsf{P}_{\mu_0}}\right)Log\left(\frac{d\mathsf{P}_\mu}{d\mathsf{P}_{\mu_0}}\right)\right] = \frac{1}{2}\left(\frac{\mu-\mu_0}{\sigma}\right)^2 T.$$

Remark *Obviously, it is minimum for* $\mu = \mu_0$. *Here, this a quadratic function of the difference between the two parameters. Note that it also decreasing with respect to volatility parameter* σ .

Proposition *For the Log Entropy case, the optimal solution can be determined by minimizing*

$$\frac{(V_0 e^{rT})^{1-\phi}}{1-\phi} E\left[\exp\left(\frac{1}{2}\theta^2 T\frac{1-\phi}{\phi}\right)\right] + \frac{1}{2}\xi\left(\frac{\mu-\mu_0}{\sigma}\right)^2 T.$$

Conclusion

Using the ambiguity theory, the optimal payoff for a given wealth can be determined for a large class models. The optimal portfolio profile proves that derivatives instruments have to be introduced into the portfolio in order to maximize the utility of the investor. The optimal solution depends clearly on the risk aversion of the investor, on his ambiguity about probability distributions and, under insurance constraint, it also depends on the insured proportion at maturity. The optimal portfolio can be determined for several forms of utility functions, ambiguity and insurance constraints. In the case without constraint, the concavity/convexity and monotonicity of portfolio profile is determined from the behavioral parameters (degree of risk aversion and ambiguity) and from the performance of financial markets, for example the Sharpe ratio type. They can be extended to continuous time models, by using for example the dynamic completeness. Assuming that dynamic ambiguity can be based on conditional expectation of the densities of the probability distributions, the optimal solution would be determined. For the static case, the observation is only relevant at the initial time, so that this type of problem does not occur.

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